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Motion of the mantle in the translational modes of the Earth and Mercury

Pavel Grinfeld*, Jack Wisdom

Department of Earth, Atmospheric, and Planetary Sciences, Massachusetts Institute of Technology, 77 Massachusetts Ave., Cambridge, MA 02139, USA

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Abstract

Slichter modes refer to the translational motion of the inner core with respect to the outer core and the mantle [Slichter, L., 1961. The fundamental free mode of the Earth's inner core. [Proc. Natl. Acad. Sci. U.S.A. 47, 186–190]. The polar Slichter mode is the motion of the inner core along the axis of rotation. Busse [Busse, F.H., 1974. On the free oscillation of the Earth's inner core. J. Geophys. Res. 79, 753–757] presented an analysis of the polar mode which yielded an expression for its period. Busse's analysis included the assumption that the mantle was stationary. This approximation is valid for planets with small inner cores, such as the Earth whose inner core is about 1/60 of the total planet mass. On the other hand, many believe that Mercury's inner core may be enormous. If so, the motion of the mantle should be expected to produce a significant effect.

We present a formal framework for including the motion of the mantle in the analysis of the translational motion of the inner core. We analyze the effect of the motion of the mantle on the Slichter modes for a non-rotating planet with an inner core of arbitrary size. We omit the effects of viscosity in the outer core, magnetic effects, and solid tides. Our approach is perturbative and is based on a linearization of Euler's equations for the motion of the fluid and Newton's second law for the motion of the inner core. We find an analytical expression for the period of the Slichter mode. Our result agrees with Busse's in the limiting case of a small inner core. We present the unexpected result that even for Mercury the motion of the mantle does not significantly change the period of the oscillation.

Keywords: Fluid dynamics; Inner core; Mantle; Slichter modes; Perturbation theory

1. Introduction

The Slichter modes, first studied in Slichter (1961), refer to the translational motion of the inner core with respect to the outer core and the mantle. The first analytical treatment of the problem was performed by Busse (1974). In the past 12 years, there has been an explosion of scientific activity in the research of the normal modes of the Earth primarily due to the emergence of superconducting gravimeters capable of detecting the relative motion of the inner core by measuring the variations in the Earth's gravitational field. Some authors

* Corresponding author.
E-mail addresses: pg@mit.edu (P. Grinfeld); wisdom@poincare.mit.edu (J. Wisdom).

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believe that evidence of Slichter modes can be found in the existing gravimeter data (Smylie, 1992). However, a definitive detection has proven to be controversial (Crossley et al., 1992; Jensen et al., 1995; Hinderer et al., 1995) and the search for Slichter modes continues today (Courtier et al., 2000; Rosat et al., 2003), along with active theoretical research (Smylie and McMillan, 2000; Rogister, 2003).

The upcoming observations of Mercury (Peale et al., 2002; Spohn et al., 2001) will present an exciting new opportunity for the study of the Slichter modes. Many believe that much like the Earth, Mercury too has a solid inner core in a fluid outer core (Schubert et al., 1988). However, unlike the Earth's inner core whose mass is 1/60 of the total planet, Mercury's inner core may be enormous (Siegfried and Solomon, 1974) with a radius that is nearly 2/3 of the radius of the planet and mass that is about 60% of the total planet. A simple conservation of linear momentum computation shows that the amplitude of the mantle's oscillations equals about 10% of the amplitude of the inner core, compared to 0.1% for the Earth. The change in the gravitational field due to a displacement of the inner core will be nearly 100 times greater for Mercury than for Earth making the detection of the modes a less challenging proposition.

The rotation of the planet leads to a split in the spectrum yielding the "Slichter triplet": a single axial (or *polar*) mode and two equatorial modes, retrograde and prograde. The three eigenperiods are evenly spaced by about a quarter of an hour. Of the three modes, the polar mode is best suited for analytical treatment. A first complete analytical treatment of the polar mode was presented by Busse (1974). Busse considered a simple three-layer model of the rotating spherical Earth, in which the outer core was incompressible, inviscid and of constant density. As we show below, Busse's results are consistent with the assumption that the outer mantle is stationary.

We generalize Busse's analysis to a planet whose outer mantle is allowed to move. In the process, we build a formal analytical framework for analyzing a three-layer planet, which can be used to incorporate more complicated effects, such as ellipticity of the layers, possible phase transformations at the inner core–outer core boundary, and compressibility of the outer core. Our analysis will be performed from "first principles": Newton's second law for the motion of the inner core and the mantle and Euler's equations for the motion of the liquid core.

Looking ahead, we find that the motion of the mantle will have little effect on the period of the Slichter modes for the Earth and even for Mercury, for which it will introduce a correction of only about 0.1%. This is an unexpected and counter-intuitive result given the significant amplitude of the mantle's oscillation. Compare two simple systems, in one a mass m is connected to a stationary wall by a spring of stiffness k. This system is analogous to a stationary mantle and it frequency of oscillation is $\sqrt{k/m}$. In the other system, the mass m is connected by the same spring to a mass of $10m$ which is free to oscillate. This is analogous to a moving mantle of a moving mantle and we explain below why $10m$ is appropriate. The frequency of oscillation of the second system is $\sqrt{11k/10m}$ which constitutes an almost 5% difference from the first system. The fact that we obtain an estimate of 0.1%, rather than 5%, highlights the effect of the fluid on the dynamics of the system and demonstrates the necessity for a formal analytical approach.

2. Model and methodology

Undoubtedly, an advanced model of the Earth is needed for a thorough analysis of the Slichter oscillations and accurate prediction of the eigenperiods. In all likelihood, such models will require the use of numerical methods. We set a more modest goal for ourselves and that is to study the effect of the motion of the mantle. We therefore choose to study a simple problem that can be carried through analytically.

We consider a three layer model of a planet (Fig. 1) with a rigid inner core Ω_1 of density ρ_1 and radius R_1, a fluid outer core Ω_2 of density ρ_2 and radius R_2, and a rigid outer mantle Ω_3 of density ρ_3 and radius R_3. We assume that each density is constant. Let S_n be the boundary of domain Ω_n. We study the oscillations of the inner core under the influence of gravity and fluid pressure. We assume that the fluid outer core is incompressible and neglect the effects of viscosity. We *exclude* from consideration the *rotation* of the planet which can affect the frequency of even the polar Slichter mode since it affects the dynamics of the fluid.

The fluid plays two important roles. It creates the restoring gravitational force and is also responsible for

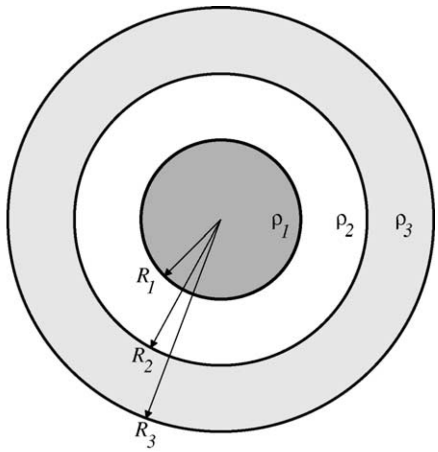

Fig. 1. The equilibrium configuration.

the effects of pressure. Our approach treats pressure in a formal way as governed by the Euler equations. Slichter and Busse provide an insightful decomposition of pressure force into a hydrostatic component and one that is a response to the acceleration of the inner core. Slichter combined the gravitational force and the hydrostatic pressure into a single expression that vanishes when the densities of the inner and outer cores match.

The spherically symmetric mantle does not exert any gravitational forces on the internal bodies. Only its total mass is relevant and so we expect that the final answer includes ρ_3 and R_3 only in the combination $\rho_3(R_3^3 - R_2^3)$. In the absence of the fluid outer core, the inner core would be in a state of neutral equilibrium (any deviation from our idealization will cause the inner core to "fall" onto the mantle). Consequently, our eventual expression for the oscillation frequency must approach zero in the limit $\rho_2 = 0$.

Our approach is perturbative, in which every configuration of the system is treated as a small deviation from the spherically symmetric stable configuration and all velocities are small. The "unperturbed" spherically symmetric gravitational potential is ψ_0 and its rate of change $\partial\psi/\partial t$, determined perturbatively, is induced by the motion of the system. We solve for the fluid velocity field v^R, v^Θ ($v^\Phi = 0$) and pressure p consistent with the translational motion of the inner core and the mantle.

We would like most of our intermediate expressions to generalize to the case of multilayer fluid planets as well as to the study of phase transitions. To this end we describe the motion of the constituents of the system by specifying the normal velocities C of the interfaces (see Appendix A) and assuming, for as long as possible, that C is completely arbitrary. We express the three normal velocity fields in the vicinity of the equilibrium configuration as harmonic series

$$C_1(\theta, \phi) = R_1\,\mathrm{i}\omega\,\mathrm{e}^{\mathrm{i}\omega t}\sum_{l,m} C_1^{lm} Y_{lm}(\theta, \phi) \tag{1a}$$

$$C_2(\theta, \phi) = R_2\,\mathrm{i}\omega\,\mathrm{e}^{\mathrm{i}\omega t}\sum_{l,m} C_2^{lm} Y_{lm}(\theta, \phi) \tag{1b}$$

$$C_3(\theta, \phi) = R_3\mathrm{i}\omega\,\mathrm{e}^{\mathrm{i}\omega t}\sum_{l,m} C_3^{lm} Y_{lm}(\theta, \phi) \tag{1c}$$

where ϕ is the longitude, θ the colatitude, and $Y_{lm}(\theta, \phi)$ are spherical harmonics normalized to unity:

$$\int_{|r|=1} Y_{l_1m_1}(\theta, \phi) Y^*_{l_2m_2}(\theta, \phi)\,\mathrm{d}S = \delta_{l_1l_2}\delta_{m_1m_2},$$

where $*$ means complex conjugation. Here and in all subsequent expressions involving complex numbers taking the real part is implied.

The normal boundary velocity of a rigid sphere can be fully represented by the $l = 1$ harmonics. If the inner core is moving with velocity $\mathbf{v} = \mathrm{i}\omega R_1\,\mathrm{e}^{\mathrm{i}\omega t}(A_1^X, A_1^Y, A_1^Z)$ then the resulting normal velocity C is given by

$$\begin{aligned} C_1(\theta, \phi) &= \mathbf{v}\cdot\mathbf{N} \\ &= v_1 \sin\theta\cos\phi + v_2\sin\theta\sin\phi + v_3\cos\theta \\ &= \mathrm{i}\omega R_1\,\mathrm{e}^{\mathrm{i}\omega t}\left(\sqrt{\frac{2\pi}{3}}(A_1^X - \mathrm{i}A_1^Y)Y_{1(-1)}(\theta, \phi)\right. \\ &\quad + \sqrt{\frac{4\pi}{3}}A_1^Z Y_{10}(\theta, \phi) \\ &\quad \left. + \sqrt{\frac{2\pi}{3}}(A_1^X + iA_1^Y)Y_{11}(\theta, \phi)\right) \end{aligned}$$

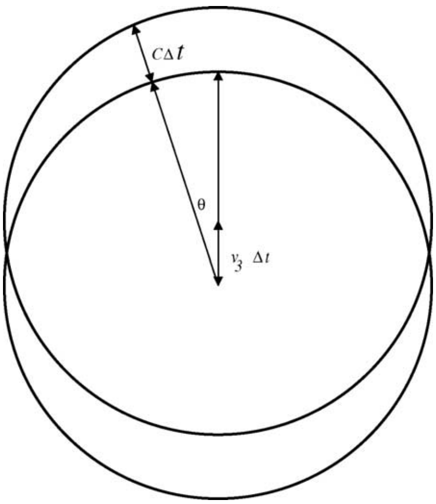

Fig. 2. The velocity of the interface induced by the velocity of the sphere's center, $C = v_3 \cos\theta$.

resulting in

$$C_1^{1(-1)} = (C^{1,1})^* = \sqrt{\frac{2\pi}{3}}(A_1^X - \mathrm{i}A_1^Y) \quad (2a)$$

$$C_1^{10} = \sqrt{\frac{4\pi}{3}}A_1^Z \quad (2b)$$

Therefore, if the oscillation of the inner core takes place in the axial direction with amplitude $R_1 A_1^Z$, Eq. (2b) represents the relationship between the amplitude and C_1^{10}. Fig. 2 illustrates this computation for the axial motion of the sphere.

3. Overview of the analysis

We consider the "axial" mode in which the inner core and the mantle oscillates along the planet's axis of rotation. Of course, since we ignore the effects of rotation, the axial direction is in no way preferred and our oscillation mode is triply degenerate. However, the degeneracy is not much of an issue since it is removed from the mathematical analysis by a priori stating the direction in which the oscillation takes place.

The oscillation frequency ω will be determined from the relationship that states that a homogeneous linear system of equations has a nonzero solution if and only if its determinant vanishes. The two equations arise from conservation of momentum for the planet as a whole and Newton's second law for the inner core. The two unknowns C_1^{10} and C_2^{10} are, essentially, the amplitudes of oscillation of the inner core and the mantle.

We nondimensionalize many of our expressions by using a length scale R_* and a density ρ_*. The particular choice of R_* and ρ_* can be made later. Introduce the dimensionless densities ϱ_n and dimensionless radii Q_n:

$$\varrho_n = \frac{\rho_n}{\rho_*}; \qquad Q_n = \frac{R_n}{R_*} \quad (3)$$

and a convenient quantity (G is the gravitational constant)

$$\Psi_* = \frac{4\pi}{3} G\rho_* R_*^2 \quad (4)$$

that has dimensions of gravitational potential.

3.1. Conservation of momentum

Conservation of momentum is particularly easy to convert to a linear equation. It is equivalent to stating the center of mass of the system remains at rest:

$$\frac{4\pi}{3}(\rho_1 - \rho_2)R_1^3 \mathrm{i}\omega R_1 A_1^Z \,\mathrm{e}^{\mathrm{i}\omega t} + \frac{4\pi}{3}\rho_2(R_2^3 - R_1^3)\mathrm{i}\omega R_2 A_2^Z \,\mathrm{e}^{\mathrm{i}\omega t} + \frac{4\pi}{3}\rho_3(R_3^3 - R_2^3)\mathrm{i}\omega R_3 A_3^Z \,\mathrm{e}^{\mathrm{i}\omega t} = 0$$

Utilizing (2b), recognizing the fact that for a rigid mantle $R_2 A_2^Z = R_3 A_3^Z$ (or $Q_2 C_2^{10} = Q_3 C_3^{10}$), nondimensionalizing by dividing through by $\rho_* R_*^4$ and, finally, cancelling $\sqrt{4\pi/3}\mathrm{i}\omega \,\mathrm{e}^{\mathrm{i}\omega t}$, we obtain our first equation:

$$(\varrho_1 - \varrho_2)Q_1^4 C_1^{10} + Q_2(\varrho_2 Q_2^3 + \varrho_3(Q_3^3 - Q_2^3))C_2^{10} = 0 \quad (5)$$

The ratio of the amplitudes of the motions of the mantle and the inner core is

$$\left|\frac{Q_2 C_2^{10}}{Q_1 C_1^{10}}\right| = \frac{(\varrho_1 - \varrho_2)Q_1^3}{\varrho_2 Q_2^3 + \varrho_3(Q_3^3 - Q_2^3)} \quad (6)$$

This quantity is small for the Earth (0.1%) but may be quite substantial for Mercury (10%). The fact that the ratio of amplitudes is about 10% explains why we chose to attach a mass of m to a mass of $10m$ in our spring example above.

3.2. Newton's second law for the inner core

The inner core experiences two forces: the gravitational force exerted at every point inside the inner core and the hydrodynamic force applied at the boundary. The gravitational force is proportional to the density of the inner core and the vector gradient of the gravitational potential ψ. The hydrostatic force is proportional to the pressure p at the boundary and points along the boundary normal. Therefore, Newton's second law reads

$$M_1\mathbf{a} = -\int_{\Omega_1} \rho_1 \nabla\psi \,\mathrm{d}\Omega - \int_{S_1} p\mathbf{N}\,\mathrm{d}S, \tag{7}$$

where $\mathbf{a}$ is the acceleration of the inner core ($a = -\omega^2 R_1 A^Z\,\mathrm{e}^{\mathrm{i}\omega t}$) and $\mathbf{N}$ is the *outward* normal—thus the minus sign for the pressure contribution.

This equation is nonlinear since the domain of integration and the integrand are both time dependent. The equation is linearized by taking a time derivative and keeping first order terms. Differentiation of integrals makes use of the following formulas for the time dependent volume and surface integrals:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega f(t,\Omega)\,\mathrm{d}\Omega = \int_\Omega \frac{\partial f(t,\Omega)}{\partial t}\,\mathrm{d}\Omega + \int_{\partial\Omega} Cf(t,\Omega)\,\mathrm{d}\Omega \tag{8a}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_S f(t,S)\,\mathrm{d}S = \int_S \frac{\delta f(t,S)}{\delta t}\,\mathrm{d}S - \int_S C\kappa f(t,S)\,\mathrm{d}S, \tag{8b}$$

where C is the invariant velocity of the interface introduced above, κ is the mean curvature of the interface, and $\delta/\delta t$ is the derivative with respect to the motion of the interface discussed briefly in Appendix A and more thoroughly in Grinfeld (2003) and Grinfeld and Wisdom (2005).

We represent the time-dependent quantities $\psi(t)$ and $p(t)$ as

$$\psi(t) = \psi_0 + \bar{\psi}(t) \tag{9a}$$

$$p(t) = p_0 + \bar{p}(t) \tag{9b}$$

and ψ_0 and p_0 are time-independent gravitational potential and pressure that correspond to the equilibrium configuration in which the inner core rests at the center of mass of the planet and $\bar{\psi}(t)$ and $\bar{p}(t)$ are small time-dependent corrections. The equilibrium gravitational potential ψ_0 and pressure p_0 satisfy the hydrostatic equation in the outer core:

$$\nabla p_0 + \rho_2 \nabla\psi_0 = 0. \tag{10}$$

An application of Eqs. (8a) and (8b) to Newton's second law (7) yields:

$$M_1\frac{\mathrm{d}\mathbf{a}}{\mathrm{d}t} = -\int_{S_1}\left(\left(\rho_1\frac{\partial\psi}{\partial t} + \frac{\partial p}{\partial t}\right)\mathbf{N} + C_1(\rho_1\nabla\psi_0 + \nabla p_0)\right)\mathrm{d}S \tag{11}$$

We make use of the equation of hydrostatic equilibrium (10) to eliminate p_0:

$$M_1\frac{\mathrm{d}\mathbf{a}}{\mathrm{d}t} = -\int_{S_1}\mathrm{d}S\left(\left(\rho_1\frac{\partial\psi}{\partial t} + \frac{\partial p}{\partial t}\right)\mathbf{N} + C_1(\rho_1-\rho_2)\nabla\psi_0\right) \tag{12}$$

Finally, we convert the vector equation into a scalar one by projecting it onto the oscillation axis by dotting the last equation with $\hat{z}$:

$$M_1\frac{\mathrm{d}a}{\mathrm{d}t} = -\int_{S_1}\mathrm{d}S\left(\rho_1\frac{\partial\psi}{\partial t} + \frac{\partial p}{\partial t} + C_1(\rho_1-\rho_2)\frac{\partial\psi_0}{\partial r}\right) \times\sqrt{\frac{4\pi}{3}}Y_{10}(\theta,\phi), \tag{13}$$

since $\mathbf{N}\cdot\hat{z} = \cos\theta$ and $\nabla\psi_0\cdot\hat{z} = \partial\psi_0/\partial z = (\partial\psi_0/\partial r)(\partial r/\partial z) = (\partial\psi_0/\partial r)\cos\theta$ and $\sqrt{4\pi/3}Y_{10}(\theta,\phi)$ was substituted for $\cos\theta$. Therefore, the three quantities to be determined are $\frac{\partial\psi_0}{\partial r}$, $\frac{\partial\psi}{\partial t}$ and $\frac{\partial p}{\partial t}$. This is the task that we are turning to now, starting with the gravitational potential.

4. Gravitational potential

This section contains an outline of how we compute the gravitational potential and its evolution. A detailed description of the method of analysis can be found in Grinfeld and Wisdom (2005).

The gravitational potential $\psi(r,\theta,\phi)$ satisfies the Poisson equation

$$\nabla^2\psi = 4\pi G\rho, \tag{14}$$

where ρ is taken to be ρ_1, ρ_2, ρ_3 or 0 depending on the region. ψ is finite at the origin, vanishes at infinity, and is continuous along with its derivatives across all interfaces. Using the notation $[X]_n$ to indicate the jump of the quantity X across the interface n (e.g. $[\rho]_1 = \rho_1 - \rho_2$), we write the continuity conditions as

$$[\psi]_1, [\psi]_2, [\psi]_3 = 0 \tag{15a}$$

$$\mathbf{N}\cdot[\nabla\psi]_1, \mathbf{N}\cdot[\nabla\psi]_2, \mathbf{N}\cdot[\nabla\psi]_3 = 0 \tag{15b}$$

The equilibrium potential $\psi_0(r)$ is straightforward to compute:

$$\psi_0(r,\theta,\phi) = \Psi_* \begin{cases} \frac{\varrho_1}{2}\left(\frac{r}{R_*}\right)^2 + A_1, & \text{inner core} \\ \frac{\varrho_2}{2}\left(\frac{r}{R_*}\right)^2 + A_2 + B_2\left(\frac{r}{R_*}\right)^{-1}, & \text{outer core} \\ \frac{\varrho_3}{2}\left(\frac{r}{R_*}\right)^2 + A_3 + B_3\left(\frac{r}{R_*}\right)^{-1}, & \text{mantle} \\ B_4\left(\frac{r}{R_*}\right)^{-1}, & \text{outside} \end{cases}, \tag{16}$$

where

$$\begin{bmatrix} A_1 \\ A_2 \\ A_3 \\ B_2 \\ B_3 \\ B_4 \end{bmatrix} = \begin{bmatrix} -\frac{3}{2}[\varrho]_1 Q_1^2 - \frac{3}{2}[\varrho]_2 Q_2^2 - \frac{3}{2}[\varrho]_3 Q_3^2 \\ -\frac{3}{2}[\varrho]_2 Q_2^2 - \frac{3}{2}[\varrho]_3 Q_3^2 \\ -\frac{3}{2}[\varrho]_3 Q_3^2 \\ -[\varrho]_1 Q_1^3 \\ -[\varrho]_1 Q_1^3 - [\varrho]_2 Q_2^3 \\ -[\varrho]_1 Q_1^3 - [\varrho]_2 Q_2^3 - [\varrho]_3 Q_3^3 \end{bmatrix} \tag{17}$$

The quantity $\partial\psi_0/\partial r$ at S_1 is given by

$$\left.\frac{\partial\psi_0}{\partial r}\right|_{S_1} = \frac{\Psi_*}{R_*}\varrho_1 Q_1 \tag{18}$$

The equations for the potential perturbation $\partial\psi/\partial t$ are obtained by differentiating the gravitational system (14)–(15b) with respect to time. The differentiation of the bulk Eq. (14) yields

$$\nabla^2\frac{\partial\psi}{\partial t} = 0 \tag{19}$$

indicating that $\partial\psi/\partial t$ is harmonic. The boundary conditions (15a) and (15b) are differentiated in the invariant sense discussed in Grinfeld (2003). We obtain that $\partial\psi/\partial t$ is continuous across all interfaces, while the normal derivative of $\partial\psi/\partial t$ jumps by an amount proportional to the velocity of the interface C and the jump in the second normal derivative of the unperturbed potential ψ_0. We use indicial notation with summation convention:

$$\left[\frac{\partial\psi}{\partial t}\right]_{1,2,3} = 0 \tag{20a}$$

$$N^i\left[\nabla_i\frac{\partial\psi}{\partial t}\right]_{1,2,3} = -C_{1,2,3}N^iN^j[\nabla_i\nabla_j\psi_0]_{1,2,3} \tag{20b}$$

These equations are obtained by applying the $\delta/\delta t$-derivative to the boundary conditons (15a) and (15b) and utilizing the algebraic properties of the $\delta/\delta t$-derivative outlined in the Appendix A.

Finally, $\partial\psi/\partial t$ is finite at the origin and vanishes at infinity. The resulting system is solved by separation of variables:

$$\frac{\partial\psi}{\partial t}(r,\theta,\phi) = \Psi_* \sum_{l,m} \begin{cases} A_1^{lm}\left(\frac{r}{R_*}\right)^l, & \text{inner} \\ A_2^{lm}\left(\frac{r}{R_*}\right)^l + B_2^{lm}\left(\frac{r}{R_*}\right)^{-l-1}, & \text{outer} \\ A_3^{lm}\left(\frac{r}{R_*}\right)^l + B_3^{lm}\left(\frac{r}{R_*}\right)^{-l-1}, & \text{mantle} \\ B_4^{lm}\left(\frac{r}{R_*}\right)^{-l-1}, & \text{outside} \end{cases} |Y_{lm}(\theta,\phi)\mathrm{i}\omega\,\mathrm{e}^{\mathrm{i}\omega t}, \tag{21}$$

The remaining sets of six coefficients are determined by satisfying the six boundary conditions (20a) and (20b). Since both sides are expressed as series in spherical harmonics, the boundary conditions are met by satisfying the identities for each of the spherical harmonics. This leads to a 6×6 linear system whose solution is

$$\begin{bmatrix} A_1^{lm} \\ A_2^{lm} \\ A_3^{lm} \\ B_2^{lm} \\ B_3^{lm} \\ B_4^{lm} \end{bmatrix} = -\frac{3}{2l+1} \begin{bmatrix} Q_1^{-l+1} & Q_2^{-l+1} & Q_3^{-l+1} \\ 0 & Q_2^{-l+1} & Q_3^{-l+1} \\ 0 & 0 & Q_3^{-l+1} \\ Q_1^{l+2} & 0 & 0 \\ Q_1^{l+2} & Q_2^{l+2} & 0 \\ Q_1^{l+2} & Q_2^{l+2} & Q_3^{l+2} \end{bmatrix} \times \begin{bmatrix} [\varrho]_1 Q_1 C_1^{lm} \\ [\varrho]_2 Q_2 C_2^{lm} \\ [\varrho]_3 Q_3 C_3^{lm} \end{bmatrix} \tag{22}$$

We would like to note that the rate of change of potential (21 and 22) applies to arbitrary perturbations of the interfaces, with simple translational motion along the axis being a special case for which $l = 1$ and $m = 0$. The presented expressions can be used to incorporate the effect of phase transformations at the inner core–outer core boundary, which may result in a very complicated evolution of the interface. Further, if the quantity t is not interpreted as time, but simply as a parameter describing the perturbation, (21 and 22) can be used to compute the corrections to the gravitational potential for near-spherical geometries. Grinfeld and Wisdom (2005) use this formula, along with its second-order counterpart, to compute the potential inside a slightly ellipsoidal cavity.

For simple translational motion, the sole present harmonic is the $l = 1$, $m = 0$ term and the expressions reduce to

$$\begin{bmatrix} A_1^{10} \\ A_2^{10} \\ A_3^{10} \\ B_2^{10} \\ B_3^{10} \\ B_4^{10} \end{bmatrix} = - \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \\ Q_1^3 & 0 & 0 \\ Q_1^3 & Q_2^3 & 0 \\ Q_1^3 & Q_2^3 & Q_3^3 \end{bmatrix} \begin{bmatrix} [\varrho]_1 Q_1 C_1^{10} \\ [\varrho]_2 Q_2 C_2^{10} \\ [\varrho]_3 Q_3 C_3^{10} \end{bmatrix} \tag{23}$$

As discussed above, the scalars C_1^{10}, C_2^{10}, and C_3^{10} are essentially the nondimensionalized amplitudes of oscillations (save for a multiplier of $\sqrt{4\pi/3}$). Of particular interest below are the values of $\partial\psi/\partial t$ in the domain $\Omega_2 - \Omega_1$ occupied by the outer core where the fliud equations are solved and the contributions to Newton's second law are made. For future reference, we present the expression for $\partial\psi/\partial t$ (once again using the fact that $Q_2 C_2^{10} = Q_3 C_3^{10}$)

$$\left.\frac{\partial\psi}{\partial t}(r,\theta,\phi)\right|_{\Omega_2-\Omega_1} = -\Psi_*\left(([\varrho]_2 + [\varrho]_3) Q_2 C_2^{10}\left(\frac{r}{R_*}\right) + [\varrho]_1 Q_1^4 C_1^{10}\left(\frac{r}{R_*}\right)^{-2}\right) Y_{10}(\theta,\phi)\mathrm{i}\omega\,\mathrm{e}^{\mathrm{i}\omega t} \tag{24}$$

At the inner core boundary, we have

$$\left.\frac{\partial\psi}{\partial t}(\theta,\phi)\right|_{S_1} = -\Psi_* Q_1(([\varrho]_2 + [\varrho]_3) Q_2 C_2^{10} + [\varrho]_1 Q_1 C_1^{10}) Y_{10}(\theta,\phi)\mathrm{i}\omega\,\mathrm{e}^{\mathrm{i}\omega t}$$

We have computed two of the three unknown quantities in the linearized Newton's law (13) and now turn our attention to the hydrodynamic pressure p.

5. Motion of the fluid

We assume that the outer core is inviscid and incompressible of constant density ρ_2. The velocity and pressure fields $\mathbf{v}$ and p are governed by Euler equations

$$\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} = -\frac{1}{\rho_2} \nabla p - \nabla \psi \tag{25a}$$

$$\nabla \cdot \mathbf{v} = 0, \tag{25b}$$

We allow slippage at the boundaries—the normal component of the velocity of the fluid matches normal velocity of the rigid interface.

The linearization procedure starts by introducing the small perturbations $\bar{\mathbf{v}}$ and $\bar{p}$ to the velocities and the pressure:

$$\mathbf{v}(t; r, \theta, \phi) = \mathbf{v}_0(r, \theta, \phi) + \bar{\mathbf{v}}(t, r, \theta, \phi) \tag{26a}$$

$$p(t; r, \theta, \phi) = p_0(r, \theta, \phi) + \bar{p}(t, r, \theta, \phi) \tag{26b}$$

The equilibrium velocities $\mathbf{v}_0$ vanish and, if we take a time derivative of the Euler equations, the unperturbed pressure p_0 will drop out as well. Therefore, away from the boundaries the motion of the fluid is governed by

$$\frac{\partial^2 \bar{\mathbf{v}}}{\partial t^2} + \frac{\partial \bar{\mathbf{v}}}{\partial t} \cdot \nabla \bar{\mathbf{v}} + \bar{\mathbf{v}} \cdot \nabla \frac{\partial \bar{\mathbf{v}}}{\partial t} = -\frac{1}{\rho} \nabla \frac{\partial \bar{p}}{\partial t} - \nabla \frac{\partial \psi}{\partial t} \tag{27a}$$

$$\nabla \cdot \bar{\mathbf{v}} = 0 \tag{27b}$$

Since $\bar{v}$ is considered small, we neglect the quadratic terms

$$\frac{\partial^2 \bar{\mathbf{v}}}{\partial t^2} = -\frac{1}{\rho_2} \nabla \frac{\partial \bar{p}}{\partial t} - \nabla \frac{\partial \psi}{\partial t} \tag{28a}$$

$$\nabla \cdot \bar{\mathbf{v}} = 0. \tag{28b}$$

Apply the divergence operator to the Euler equations (28a)

$$\frac{\partial^2 \nabla \cdot \bar{\mathbf{v}}}{\partial t^2} = -\frac{1}{\rho} \nabla^2 \frac{\partial \bar{p}}{\partial t} - \nabla^2 \frac{\partial \psi}{\partial t} \tag{29a}$$

Since the fluid is incompressible (28b) and $\partial\psi/\partial t$ is harmonic (19) we observe that $\partial\bar{p}/\partial t$ is harmonic. Therefore, in attempting to solve the linearized hydrodynamic system by separation of variables, we know to use form (31) for the radial part of $\bar{p}$. Being able to make this guess saves us the trouble of solving a system of ODE's.

We arrive at the following solution for the pressure correction $\bar{p}$ and velocities $\bar{v}$:

$$\bar{p}(t, r, \theta, \phi) = \rho_2 R_*^2 \omega^2 P^{lm}(r) Y_{lm}(\theta, \phi)\, \mathrm{e}^{\mathrm{i}\omega t} \tag{30a}$$

$$\bar{v}^R(t, r, \theta, \phi) = R_* \omega F^{lm}(r) Y_{lm}(\theta, \phi)\, \mathrm{e}^{\mathrm{i}\omega t} \tag{30b}$$

$$\bar{v}^\Theta(t, r, \theta, \phi) = \frac{R_*^2 \omega}{r^2} G^{lm}(r) \frac{\partial Y_{lm}(\theta, \phi)}{\partial \theta}\, \mathrm{e}^{\mathrm{i}\omega t} \tag{30c}$$

$$\bar{v}^\Phi(t, r, \theta, \phi) = 0 \tag{30d}$$

where

$$P^{lm}(r) = d_+^{lm} \left(\frac{r}{R_*}\right)^l + d_-^{lm} \left(\frac{r}{R_*}\right)^{-l-1} \tag{31}$$

$$F^{lm}(r) = \mathrm{i} R_* \frac{\mathrm{d}P^{lm}(r)}{\mathrm{d}r} \tag{32}$$

$$G^{lm}(r) = P^{lm}(r) \tag{33}$$

and the coefficients d_+^{lm} and d_-^{lm} are determined by the slippage boundary conditions which lead to the following system:

$$\begin{bmatrix} d_+^{lm} \\ d_-^{lm} \end{bmatrix} = \begin{bmatrix} l & -(l+1) \\ l Q_2^{l-1} & -(l+1) Q_2^{-l-2} \end{bmatrix}^{-1} \begin{bmatrix} C_1^{lm} \\ C_2^{lm} \end{bmatrix} - \frac{\Psi_*}{\omega^2 R_1^2} \begin{bmatrix} A_2^{lm} \\ B_2^{lm} \end{bmatrix} \tag{34}$$

The expression for pressure (30a) may be multiplied by the normal and integrated over the outer core boundary to yield the total hydrodynamic force acting on the inner core.

We now substitute Eqs. (17), (22) and (34) and $Q_2 C_2^{10} = Q_3 C_3^{10}$ (the mantle's boundaries move together) into Newton's second law (13) to obtain the second algebraic equations for C_1^{10} and C_2^{10}:

$$\left(\frac{2(Q_2^3 - Q_1^3)(\varrho_1 - \varrho_2)\Psi_*}{R_*^2 \omega^2} Q_1 \varrho_2 - (3\varrho_2 Q_2^3 + 2(Q_2^3 - Q_1^3)(\varrho_1 - \varrho_2)) Q_1 \right) C_1^{10}$$

$$+\left(3\varrho_2 Q_2^4 - \frac{2(Q_2^3 - Q_1^3)(\varrho_1 - \varrho_2)\Psi_*}{R_*^2\omega^2} Q_2\varrho_2\right) C_2^{10} = 0 \quad (35)$$

As stated above, the frequency of oscillation ω is obtained by equating to zero the determinant of the system (5 and 35). If the mantle were treated as stationary as was done by Busse, Eq. (35) alone would yield ω by setting $C_2^{10} = 0$ and equating the coefficient of C_1^{10} to zero. This simple computation would lead directly to Busse's formula.

6. Expression for the frequency and analysis of results

Equating the determinant of the linear system (5), (35) to zero, we can easily solve for ω^2. The answer is most easily presented by introducing the total mass of the system M:

$$M = \frac{4\pi}{3}(\rho_1 R_1^3 + \rho_2(R_2^3 - R_1^3) + \rho_3(R_3^3 - R_2^3)) \quad (36)$$

Also introduce three more mass-like quantities

$$m = \frac{4\pi}{3}(\rho_1 - \rho_2)R_1^3 \quad (37)$$

$$D = \frac{4\pi}{3}(\rho_1 - \rho_2)(R_2^3 - R_1^3) \quad (38)$$

$$E = \frac{4\pi}{3}\rho_2 R_2^3 \quad (39)$$

Then the natural frequency of oscillation is given by

$$\omega^2 = \frac{4\pi}{3}G\rho_2 \frac{D}{\frac{3}{2}E + D(1 - m/M)} \quad (40)$$

If we consider a planet with a small inner core, $m \ll M$, we arrive at Busse's expression:

$$\omega_{m\ll M}^2 = \frac{4\pi}{3}G\rho_2 \frac{D}{\frac{3}{2}E + D} \quad (41)$$

7. Implications for the Earth and Mercury

We make the following assumptions for the Earth and Mercury (Siegfried and Solomon, 1974):

	Earth	Mercury
R_1 (10^6m)	1.2	1.7
R_2 (10^6m)	3.5	1.8
R_3 (10^6m)	6.3	2.4
ρ_1 (10^3 kg m^{-3})	13.0	9.5
ρ_2 (10^3 kg m^{-3})	12.0	8.0
ρ_3 (10^3 kg m^{-3})	4.5	3.0

Using these values, we compute the intermediate quantities:

	Earth	Mercury
M (kg)	6.07×10^{24}	3.27×10^{23}
m (10^{22} kg)	7.24×10^{21}	3.09×10^{22}
D (10^{22} kg)	1.72×10^{23}	5.77×10^{21}
E (10^{22} kg)	2.16×10^{24}	1.95×10^{23}

And finally use the formulas (40) and (41) to arrive at the following estimates for the eigenperiod, T,

	Earth	Mercury
T (h)	4.2361	8.3983
$T_{m\ll M}$ (h)	4.2359	8.3906

We find that the motion of the mantle introduces a correction for the Earth that is less than one hundredth of 1Mercury is about 0.1%. This correction is unexpectedly small given the assumed massiveness of Mercury's inner core. In fact, we have not been able to find a simple explanation for the smallness of this correction.

8. Conclusions

We have presented a general framework for analyzing the free modes of the Earth that include the rigid motion of the planet's mantle. This motion is negligible for the Earth, but for Mercury, whose inner core may constitute more than half of the planet's total weight, the motion would be significant. We solved a linearized system of Euler's equation for the fluid outer core. The motions of the inner core and mantle are governed by Newton's second law which includes the gravitational and hydrostatic effects. For Mercury, the oscillation of the mantle is significant as its amplitude is about 10% of that of the inner core. We applied our analysis to compute an analytical expression for the period of the Slichter mode for a nonrotating planet. Our expression reduces to Busse's for the case of a small inner core. We showed the surprising and counter-intuitive result that incorporating the rigid motion of the mantle has a minimal effect on the *period* of Slichter oscillations for Mercury as well as the Earth.

If the translational mode is excited and it is possible to measure the amplitude of the mantle and the change in gravitational field on the surface of Mercury, such measurements may allow a direct determination of the size of the inner core. Suppose that a gravimeter is placed at the north pole and that the translational mode is along the axis of rotation. Then if the mantle moves up by the amount A then according to Eq. (6) the mantle moves down by $B = A(\rho_2 R_2^3 + \rho_3(R_3^3 - R_2^3))((\rho_1 - \rho_2)R_1^3)^{-1}$. Thus, the gravimeter is closer to the inner core by

$$A + B = A\left(\frac{(\rho_1 - \rho_2)R_1^3 + (\rho_2 - \rho_3)R_2^3 + \rho_3 R_3^3}{(\rho_1 - \rho_2)R_1^3}\right) \tag{42}$$

The observed change in the gravitational field is due to the mass $(\rho_1 - \rho_2)R_1^3$ moving closer by the amount B. Therefore, the gravitational field, whose magnitude is $G(\rho_1 - \rho_2)R_1^3/r^2|_{r=R_3}$, will increase by $2G(\rho_1 - \rho_2)R_1^3/r^3(A + B)|_{r=R_3}$, where G is the gravitational constant. In other words, if g is the accelaration of gravity and Δg is amplitude of the observed change then

$$\Delta g = 2G((\rho_1 - \rho_2)Q_1^3 + (\rho_2 - \rho_3)(R_2/R_3)^3 + \rho_3)A \tag{43}$$

For Mercury, the coefficient of proportionality between Δg and A in terms of R_1/R_3 is $3 \times 10^3 G((R_1/R_3)^3 + 15.231)$ which, if measured experimentally, would yield the size of the inner core.

Appendix A. The $\delta/\delta\, t$-derivative

This paper relies heavily on the calculus of moving surfaces, which has an illustrious history. This appendix briefly introduces the main concepts. An in-depth discussion of the $\delta/\delta t$-derivative and its properties can be found in Grinfeld (2003) and Grinfeld and Wisdom (2005).

Consider a one parameter family of curves S_τ indexed by a time-like parameter τ. The family S_τ can also be thought of as a time evolution of a single curve S. Let $T_\tau(S_\tau)$ be a scalar field defined on S_τ, so T not only changes its values with the passing of time but also sees its domain of definition change as well.

We present a geometric definition of the $\delta/\delta\tau$-derivative at a point ξ on the surface S_τ at time τ, illustrated in Fig. 3. Consider two locations of the surface S_τ and S_{τ^*} at nearby times τ and τ^*. Draw the straight line orthogonal to S_τ passing through the point ξ mark the point ξ^* where this straight line intersects S_{τ^*}. Define:

$$\frac{\delta\, T_\tau}{\delta\, \tau} = \lim_{\tau^* \to \tau} \frac{T_{\tau^*}(\xi^*) - T_\tau(\xi)}{\tau^* - \tau} \tag{44}$$

Let $\Delta\mathbf{z}$ be the vector connecting the point ξ to the point ξ^*. Then the velocity of the interface C (also known as the *normal* velocity) is defined as

$$C = \lim_{\tau^* \to \tau} \frac{\Delta\mathbf{z} \cdot \mathbf{N}}{\tau^* - \tau}, \tag{45}$$

where $\mathbf{N}$ is the unit normal to the surface S_τ. Since by construction $\Delta\mathbf{z}$ is aligned with $\mathbf{N}$, the projection $\Delta\mathbf{z}$ onto the normal is performed largely for the purposes of determining the sign of C. If $\mathbf{z}$ is the radius vector with respect to an arbitrary origin then the definition of C (45) can be rewritten as

$$C = \frac{\delta\mathbf{z}}{\delta\tau} \cdot \mathbf{N}$$

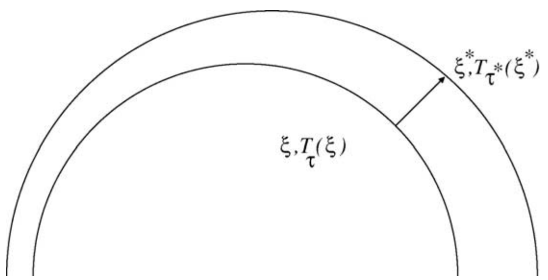

Fig. 3. Geometic definition of the $\delta/\delta t$-derivative as applied to a scalar field T_τ define at time τ on the surface S_τ with surface coordinates ξ.

The velocity field C completely determines the evolution of the interface much like the velocity field of a fluid completely determines its flow, with one significant difference. In the flow of a fluid, the trajectories of individual particles are usually of interest and the velocity field allows one to track them. The velocity C, on the other hand, describes the motion of the surface as a geometric object without keeping track of individual points.

The following equations present the key algebraic properties of the $\delta/\delta t$-derivative that are used to derive Eqs.(20a) and (15b). The quantity s is the surface shift tensor, m is the surface metric, B is the curvature tensor, ∇_S is the surface gradient and Δ_S is the surface Laplacian.

$$\frac{\delta s}{\delta \tau} = \nabla_S(CN) \tag{46a}$$

$$\frac{\delta m}{\delta \tau} = -2CB \tag{46b}$$

$$\frac{\delta N}{\delta \tau} = -s\nabla^S C \tag{46c}$$

$$\frac{\delta B}{\delta \tau} = \Delta_S C + CB^2 \tag{46d}$$

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