The Poincaré Equations

Jack Wisdom

The Poincaré equations are a generalization of the Lagrange equations. The Poincaré equations are written in terms of the quasivelocities with respect to a possibly noncommuting basis on the configuration manifold. With respect to a generalized coordinate basis the quasivelocities are the usual generalized velocities. In this case the Poincaré equations are the usual Lagrange equations. For a rigid body, using a basis that corresponds to the noncommuting rotations about the three principal axes, the Poincaré equations are the Euler equations.

Quasivelocities

The quasivelocities are defined as follows. Consider a function of coordinates h. This function on the path q is $\bar{h} = h \circ q$. Let e be a tuple of basis vectors, not necessarily commuting, on the configuration manifold N, then

$$\begin{aligned} D\bar{h}(t) &= Dh(q(t))Dq(t) \\ &= \sum_i \mathsf{e}_i(h)(q(t))\omega^i(t, q(t), Dq(t)) \end{aligned} \tag{0.1}$$

This implicitly defines the quasivelocities ω^i. They are linear functions of the generalized velocities.

The configuration path γ maps t on the real time line to the configuration manifold N. Let χ be a tuple of coordinate functions on N. So $q = \chi \circ \gamma$, is the coordinate path as a function of time. The tuple of coordinate basis vectors on N is X, and $\partial/\partial t$ is the basis vector on the time line. The basis vectors over the map γ are $\mathsf{X}_i^\gamma(\mathsf{f}) = \mathsf{X}_i(\mathsf{f}) \circ \gamma$, and dx_γ^i are the dual basis over the map. Similarly, for the basis e the basis over the map is $\mathsf{e}^\gamma(f) = \mathsf{e}(f) \circ \gamma$, and $\tilde{\mathsf{e}}_\gamma$ is the dual basis over the map: $\tilde{\mathsf{e}}_\gamma^i(\mathsf{e}_j^\gamma) = \delta_j^i$. The ordinary generalized velocities are

$$Dq^i(t) = \mathsf{dx}_\gamma^i(d\gamma(\partial/\partial t))(t), \tag{0.2}$$

where $d\gamma$ is the differential of γ. The quasivelocities are

$$\omega^i(t) = \tilde{\mathsf{e}}_\gamma^i(d\gamma(\partial/\partial t))(t). \tag{0.3}$$

The quasivelocity as a function of time can be abstracted to a state function of time, generalized coordinates, and generalized velocities.

Rotations

This section derives the basis that corresponds to rotations about the principal axes. We use these to show that the quasivelocities are the usual angular velocities about the principal axes.

The orientation of the rigid body is represented by a rotation $\mathsf{M}(q)$ from a reference orientation to the actual orientation. Along a coordinate path q at time t the orientation is $\mathsf{M}(q(t))$. Each constituent is rotated: $\xi_\alpha(t) = M(t)\xi_\alpha(0) = \mathsf{M}(q(t))\xi_\alpha(0)$. The velocity of the constituent is

$$D\xi_\alpha(t) = DM(t)\xi_\alpha(0) = DM(t)(M(t))^{-1}\xi_\alpha(t). \tag{0.4}$$

As $M(t)$ is a rotation,

$$A(\omega(t)) = DM(t)(M(t))^{-1} \tag{0.5}$$

is skew symmetric. In three dimensions, the skew-symmetric $A(\omega(t))$ can be written as the cross product with $\omega(t)$, the spatial angular velocity vector. The linear transformation $(M(t))^{-1}$ takes spatial vectors to vectors with respect to the body in its reference orientation, so the angular velocity with respect to the principal axes is $\omega'(t) = (M(t))^{-1}\omega(t)$. In terms of M,

$$A(\omega'(t)) = (M(t))^{-1}DM(t). \tag{0.6}$$

The Euler angles are local coordinates on the manifold of rotations:

$$\mathsf{M}(\theta, \phi, \psi) = R_z(\phi)R_x(\theta)R_z(\psi). \tag{0.7}$$

The basis vectors are $\partial/\partial\theta$, $\partial/\partial\phi$, and $\partial/\partial\psi$.

Next we find the basis that corresponds to rotations about the spatial rectangular coordinate axes. Given a body in the orientation specified by (θ, ϕ, ψ) the orientation after an additional ϵ rotation about the $\hat{x}$ axis is given by

$$R_x(\epsilon)R_z(\phi)R_x(\theta)R_z(\psi). \tag{0.8}$$

This corresponds to a rotation with Euler angles

$$R_z(\phi + a\epsilon)R_x(\theta + b\epsilon)R_z(\psi + c\epsilon), \tag{0.9}$$

where a, b, and c give the direction in Euler coordinates of the rotation about the $\hat{x}$ direction. Equating the two expressions

$$\begin{aligned} R_x(\epsilon) = {} & R_z(\phi + a\epsilon)R_x(\theta + b\epsilon)R_z(\psi + c\epsilon) \\ & \times (R_z(\psi))^{-1}(R_x(\theta))^{-1}(R_z(\phi))^{-1}, \end{aligned} \tag{0.10}$$

taking the derivative (with respect to ϵ), and setting $\epsilon = 0$, we derive a set of linear equations for the a, b, and c. Solving these we find

$$\begin{aligned} a &= -\sin\phi\cos\theta/(\sin\theta) \\ b &= \cos\phi \\ c &= \sin\phi/(\sin\theta). \end{aligned} \tag{0.11}$$

We conclude

$$e_x = -\frac{\sin\phi\cos\theta}{\sin\theta}\frac{\partial}{\partial\phi} + \cos\phi\frac{\partial}{\partial\theta} + \frac{\sin\phi}{\sin\theta}\frac{\partial}{\partial\psi} \tag{0.12}$$

Similarly,

$$e_y = \frac{\cos\phi\cos\theta}{\sin\theta}\frac{\partial}{\partial\phi} + \sin\phi\frac{\partial}{\partial\theta} - \frac{\cos\phi}{\sin\theta}\frac{\partial}{\partial\psi} \tag{0.13}$$

and

$$e_z = \frac{\partial}{\partial\phi}. \tag{0.14}$$

These basis vectors do not commute. The commutators satisfy

$$\begin{aligned} [e_x, e_y] &= -e_z \\ [e_y, e_z] &= -e_x \\ [e_z, e_x] &= -e_y. \end{aligned} \tag{0.15}$$

We call this the $so(3)$ basis.

To the spatial angular velocity ω there corresponds an angular velocity ω' with respect to the principal axes: $\omega = M\omega'$. Both ω

and ω' may be considered quasivelocities; we would like the basis corresponding to ω' for which equation (0.1) holds. Thus

$$e(h)(q(t))\omega(t, q(t), Dq(t)) = e'(h)(q(t))\omega'(t, q(t), Dq(t)). \tag{0.16}$$

Now, from $\omega = M\omega'$ we deduce

$$e(h)\mathsf{M} = e'(h). \tag{0.17}$$

Written out,

$$\begin{aligned}
e'_x &= \cos\psi \frac{\partial}{\partial\theta} + \frac{\sin\psi}{\sin\theta}\frac{\partial}{\partial\phi} - \frac{\sin\psi\cos\theta}{\sin\theta}\frac{\partial}{\partial\psi} \\
e'_y &= -\sin\psi \frac{\partial}{\partial\theta} + \frac{\cos\psi}{\sin\theta}\frac{\partial}{\partial\phi} - \frac{\cos\psi\cos\theta}{\sin\theta}\frac{\partial}{\partial\psi} \\
e'_z &= \frac{\partial}{\partial\psi}
\end{aligned} \tag{0.18}$$

The commutators satisfy

$$\begin{aligned}
\left[e'_x, e'_y\right] &= e'_z \\
\left[e'_y, e'_z\right] &= e'_x \\
[e'_z, e'_x] &= e'_y.
\end{aligned} \tag{0.19}$$

We call this the $so(3)$ body basis. Note the sign of the commutators relative to those of the $so(3)$ basis.

We use the $so(3)$ body basis and its dual to compute the components of the quasivelocity according to equation (0.3). The quasivelocity components are the usual body components of the angular velocities:

```
(show-expression
 (let* ((gamma
         (compose (rotations '->point)
                  (up (literal-function 'theta)
                      (literal-function 'phi)
                      (literal-function 'psi))))
        (basis-over-mu
         (basis->basis-over-map gamma so3p-basis))
        (1form-basis (basis->1form-basis basis-over-mu)))
   ((1form-basis ((differential gamma) d/dt))
    ((real-time-line '->point) 't))))
```

$$\boxed{\begin{pmatrix} \sin(\theta(t))\sin(\psi(t))\,D\phi(t) + \cos(\psi(t))\,D\theta(t) \\ \sin(\theta(t))\,D\phi(t)\cos(\psi(t)) - \sin(\psi(t))\,D\theta(t) \\ \cos(\theta(t))\,D\phi(t) + D\psi(t) \end{pmatrix}}$$

Variations

For realizable paths, the variation of the action is stationary with respect to variations of the path that leave the path fixed at the end times. The core of the derivation of the Poincaré equations is an identity linking variations, quasivelocities, and non-commutative bases.

The variation of a path function is obtained by substituting a parametric family of paths for the path and differentiating with respect to the parameter. In coordinates, a simple parametric family of paths near the path q is $\tilde{q}(\epsilon) = q + \epsilon\eta$, where the path-like η specifies the direction of the variation. The variation of $f[q]$ is

$$\delta_\eta f[q] = Dg(0) \quad \text{with} \quad g(\epsilon) = f[q + \epsilon\eta]. \tag{0.20}$$

For our purposes here it is more convenient to define variations slightly differently. Let $\tilde{q}(t, \epsilon) = q(t) + \epsilon\eta(t)$, and

$$\tilde{f}(t, \epsilon) = f[\tilde{q}(t, \epsilon)](t) \tag{0.21}$$

then

$$\delta_\eta f[q](t) = (\partial_1 \tilde{f})(t, 0). \tag{0.22}$$

Consider the path function $f[q](t) = h(q(t))$ that depends only on the coordinates of the path q at time t. Let e be a tuple of basis vectors, not necessarily commuting, on the configuration manifold N, then, using equation (0.1),

$$\partial_0 \tilde{f}(t, \epsilon) = \sum_i \mathsf{e}_i(h)(\tilde{q}(t, \epsilon))\omega^i(t, \tilde{q}(t, \epsilon), \partial_0 \tilde{q}(t, \epsilon)) \tag{0.23}$$

Analogously,

$$\partial_1 \tilde{f}(t,\epsilon) = \sum_j \mathsf{e}_j(h)(\tilde{q}(t,\epsilon)) w^j(t, \tilde{q}(t,\epsilon), \partial_1 \tilde{q}(t,\epsilon)). \tag{0.24}$$

We might call w the quasivariation.

Leaving out the arguments of $\tilde{w}$ and $\tilde{\omega}$ for brevity, the cross partials are

$$\begin{aligned}
&\partial_1 \partial_0 \tilde{f}(t,\epsilon) \\
&\quad = \sum_j \sum_i \mathsf{e}_j(\mathsf{e}_i(h))(\tilde{q}(t,\epsilon)) \tilde{\omega}^i \tilde{w}^j + \sum_i \mathsf{e}_i(h)(\tilde{q}(t,\epsilon)) \partial_1 \tilde{\omega}^i \\
&\partial_0 \partial_1 \tilde{f}(t,\epsilon) \\
&\quad = \sum_i \sum_j \mathsf{e}_i(\mathsf{e}_j(h))(t, \tilde{q}(t,\epsilon)) \tilde{w}^j \tilde{\omega}^i + \sum_j \mathsf{e}_j(h)(\tilde{q}(t,\epsilon)) \partial_0 \tilde{w}^j,
\end{aligned} \tag{0.25}$$

where

$$\begin{aligned}
\tilde{\omega}(t,\epsilon) &= \omega(t, \tilde{q}(t,\epsilon), \partial_0 \tilde{q}(t,\epsilon)) \\
\tilde{w}(t,\epsilon) &= w(t, \tilde{q}(t,\epsilon), \partial_1 \tilde{q}(t,\epsilon)).
\end{aligned}$$

The left-hand sides are equal, so we find

$$\sum_{ij} [\mathsf{e}_i, \mathsf{e}_j](h)(\tilde{q}(t,\epsilon)) \tilde{\omega}^i \tilde{w}^j = \sum_k \mathsf{e}_k(h)(\tilde{q}(t,\epsilon)) \left(\partial_1 \tilde{\omega}^k - \partial_0 \tilde{w}^k \right). \tag{0.26}$$

Assume that the basis commutators are

$$[\mathsf{e}_i, \mathsf{e}_j](h) = \sum_k c_{ij}^k \mathsf{e}_k(h), \tag{0.27}$$

then

$$\sum_{ijk} \mathsf{e}_k(f) c_{ij}^k \tilde{\omega}^i \tilde{w}^j = \sum_k \mathsf{e}_k(f)(\tilde{q}(t,\epsilon)) \left(\partial_1 \tilde{\omega}^k - \partial_0 \tilde{w}^k \right). \tag{0.28}$$

The basis elements are independent, so

$$\partial_1 \tilde{\omega}^k - \partial_0 \tilde{w}^k = \sum_{ij} \tilde{\omega}^i \tilde{w}^j c_{ij}^k. \tag{0.29}$$

We are interested in these relations along the path being varied. So we set $\epsilon = 0$. Equation (0.29) is the key relation that is needed to complete the derivation of the Poincare equations.

For a coordinate basis, the commutators are zero. The quasivelocity is just the generalized velocity. And for the variation $\tilde{q}(t, \epsilon) = q(t) + \epsilon\eta(t)$, the quasivariation is just $\eta(t)$. Relation (0.29) is then a statement about the commutation of time derivative and variation.

Derivation of the Poincaré Equations

Next we use relation (0.29) to derive the equations governing the dynamics.

The action for Lagrangian L, on the path q, from time t_0 to time t_1 is

$$S[q](t_0, t_1) = \int_{t_0}^{t_1} L(t, q(t), Dq(t))dt. \tag{0.30}$$

We reexpress the Lagrangian in terms of the quasivelocities:

$$L(t, q(t), Dq(t)) = \hat{L}(t, q(t), \omega(t, q(t), Dq(t))). \tag{0.31}$$

This defines $\hat{L}$.

The variation of the action is computed by replacing q with $\tilde{q}$ and taking the derivative with respect to the variation parameter. The integrand is then

$$\begin{aligned}&\partial_1 \hat{L}(t, \tilde{q}(t, \epsilon), \omega(t, \tilde{q}(t, \epsilon), \partial_0 \tilde{q}(t, \epsilon)))\partial_1 \tilde{q}(t, \epsilon) + \\ &\quad \partial_2 \hat{L}(t, \tilde{q}(t, \epsilon), \omega(t, \tilde{q}(t, \epsilon), \partial_0 \tilde{q}(t, \epsilon)))\partial_1 \tilde{\omega}(t, \epsilon)\end{aligned} \tag{0.32}$$

Rewriting the first term in terms of the basis e and the quasivariation w:

$$\sum_k e_k(\hat{L})\tilde{w}^k + \partial_2 \hat{L} \partial_1 \tilde{\omega} \tag{0.33}$$

where we have left out more arguments for brevity. The basis vector operates only on the second argument of $\hat{L}$. Using relation (0.29), the action integrand is

$$\sum_k \left(e_k(\hat{L})\tilde{w}^k + \partial_{2,k} \hat{L} \left(\partial_0 \tilde{w}^k + \sum_{ij} \tilde{\omega}^i \tilde{w}^j c_{ij}^k \right) \right). \tag{0.34}$$

Relabelling indices,

$$\sum_k \left(\partial_{2,k}\hat{L}\partial_0\tilde{w}^k + e_k(\hat{L})\tilde{w}^k + \sum_{ij} c^j_{ik}(\partial_{2,j}\hat{L})\tilde{\omega}^i\tilde{w}^k \right). \tag{0.35}$$

Recall that each of these terms is evaluated on the path $q(t) = \tilde{q}(t, 0)$. To simplify the notation, let $\tilde{f}(t) = \tilde{f}(t, 0)$. That is, if the second argument of a function on a varied path is zero the second argument can be omitted. The term $\partial_0\tilde{w}$ is the time derivative of w evaluated on the path q.

Let $\tilde{\Gamma}$ be the function that takes a path and gives the arguments of a Lagrangian-like function of time, coordinates, and quasivelocities

$$\tilde{\Gamma}[q](t) = (t, q(t), \omega(t, q(t), Dq(t))). \tag{0.36}$$

Returning to the action integral, we do the usual integration by parts to get

$$S[q](t_0, t_1) = \left(\partial_2\hat{L} \circ \tilde{\Gamma}[q]\right) \tilde{w}\Big|_{t_0}^{t_1} - S'[q](t_0, t_1) \tag{0.37}$$

where

$$\begin{aligned} S'[q](t_0, t_1) = \int_{t_0}^{t_1} \sum_k (D(\partial_{2,k}\hat{L} \circ \tilde{\Gamma}[q]) - e_k(\hat{L}) \circ \tilde{\Gamma}[q] \\ - \sum_{ij}(c^j_{ik} \circ q)(\partial_{2,j}\hat{L} \circ \Gamma[q])\tilde{\omega}^i)\tilde{w}^k \end{aligned} \tag{0.38}$$

Since the quasivariations are independent, and zero at the end times, stationary action implies the path satisfies the Poincaré equations:

$$D(\partial_{2,k}\hat{L} \circ \tilde{\Gamma}[q]) = e_k(\hat{L}) \circ \tilde{\Gamma}[q] + \sum_{ij}(c^j_{ik} \circ q)(\partial_{2,j}\hat{L} \circ \Gamma[q])\tilde{\omega}^i. \tag{0.39}$$

The factor $\partial_2\hat{L}$ is the momentum conjugate to the quasivelocity. The Poincaré equations state that the time derivative of this momentum along a path is equal to a derivative of $\hat{L}$ obtained by applying the corresponding basis vector plus a term consisting of products of the quasivelocities and momenta with commutator factors.

The Poincaré equations (0.39) are a second order system of differential equations. We can write them as a pair of first order equations for the generalized coordinates and the quasivelocities. Let $\sigma(t) = (t, q(t), \omega(t))$, the Poincaré equations are

$$D(\partial_{2,k}\hat{L} \circ \sigma) = e_k(\hat{L}) \circ \sigma + \sum_{ij}(c^j_{ik} \circ q)(\partial_{2,j}\hat{L} \circ \sigma)\tilde{\omega}^i. \tag{0.40}$$

The system is completed with an equation for the generalized velocities in terms of the quasivelocities. The Poincaré equations show that the equations governing the evolution of the quasivelocities form a closed subsystem whenever the Lagrangian expressed in terms of them has no coordinate dependence.

Define the structure of structure constants

```
(define ((structure-constants basis) q)
  (s:map (lambda (edualj)
           (s:map (lambda (ei)
                    (s:map (lambda (ek)
                             ((edualj (commutator ei ek)) q))
                           (basis->vector-basis basis)))
                  (basis->vector-basis basis)))
         (basis->1form-basis basis)))
```

Then define the Poincaré equations for the paths q and ω:

```
(define (((poincare-equations Lhat basis) q w) t)
  (let ((e (basis->vector-basis basis))
        (c (basis->structure-constants basis)))
    (- ((D (lambda (t)
             (((partial 2) Lhat) (up t (q t) (w t)))))
        t)
       (+ ((e (lambda (x) (Lhat (up t x (w t))))) (q t))
          (* (* c (((partial 2) Lhat) (up t (q t) (w t))))
             (w t))))))
```

For a free rigid body the Lagrangian is the kinetic energy. Written in terms of the components of the angular velocity (quasivelocity) on the principal axes the Lagrangian is

$$\hat{L}(t, q, \omega) = \frac{1}{2}(A(\omega^a)^2 + B(\omega^b)^2 + C(\omega^c)^2). \tag{0.41}$$

The momenta are $[A\omega^a, B\omega^b, C\omega^c]$. Using the body basis for $so(3)$, the Poincaré equations are

```
(show-expression
 (((poincare-equations (Lhat 'A 'B 'C) so3p-basis)
   (up (literal-function 'theta)
       (literal-function 'phi)
       (literal-function 'psi))
   (up (literal-function 'omega-a)
       (literal-function 'omega-b)
       (literal-function 'omega-c)))
  't))
```

$$\boxed{\begin{bmatrix} -B\omega^b(t)\,\omega^c(t) + C\omega^b(t)\,\omega^c(t) + AD\omega^a(t) \\ \\ A\omega^a(t)\,\omega^c(t) - C\omega^a(t)\,\omega^c(t) + BD\omega^b(t) \\ \\ -A\omega^a(t)\,\omega^b(t) + B\omega^a(t)\,\omega^b(t) + CD\omega^c(t) \end{bmatrix}}$$

These are Euler's equations for the evolution of the body components of the angular velocities. The system is completed with an equation that relates the body angular velocity components to the generalzed velocities.

$$\begin{bmatrix} D\theta \\ D\varphi \\ D\psi \end{bmatrix} = \frac{1}{\sin\theta} \begin{bmatrix} \cos\psi\sin\theta & -\sin\psi\sin\theta & 0 \\ \sin\psi & \cos\psi & 0 \\ -\sin\psi\cos\theta & -\cos\psi\cos\theta & \sin\theta \end{bmatrix} \begin{bmatrix} \omega^a \\ \omega^b \\ \omega^c \end{bmatrix}. \quad (0.42)$$

These are derived from the definition of the quasivelocities in terms of the Lagrangian state by inversion.

We can use the same procedure to evaluate the second-order Poincaré equations by writing the quasivelocity in terms of the Lagrangian state. For example, we can demonstrate that the spatial z component of the second order Poincaré equations is the same as the ϕ component of the second order Lagrange equations. Both express the conservation of the z component of the usual angular momentum, which is also the momentum conjugate to ϕ in the Lagrange equations, and the momentum conjugate to the quasivelocity ω^z in the Poincaré equations. We rewrite the Lagrangian in terms of the spatial components of the angular velocities

$$\tilde{L}(t, q, \omega) = \hat{L}(t, q, (\mathsf{M}(q))^{-1}\omega). \quad (0.43)$$

Then compute:

```
(define Euler-path
  (up (literal-function 'theta)
      (literal-function 'phi)
      (literal-function 'psi)))

(pe (- (ref (((Lagrange-equations (Lrigid 'A 'B 'C))
              Euler-path) 't)
            1)
       (ref (((poincare-equations (Ltilde 'A 'B 'C)
                                  so3-basis)
              Euler-path
              (compose Euler-state->omega
                       (Gamma Euler-path)))
             't)
            2)))
0
```

We have used the $so(3)$ basis, for which the spatial angular velocities are the quasivelocities.

For a coordinate basis, the quasivelocity is the usual generalized velocity, and the momentum is the usual generalized momentum. The commutators are zero, and the basis vector applied to the Lagrangian is just the coordinate derivative of the Lagrangian. So the Poincaré equations then reduce to the Lagrange equations. For example:

```
(pe
 (- (((Lagrange-equations (Lrigid 'A 'B 'C))
      Euler-path) 't)
    (((poincare-equations (Lrigid 'A 'B 'C)
                          rotations-basis)
      Euler-path
      (D Euler-path))
     't)))
(down 0 0 0)
```

Hamiltonian Formulation

The Poincaré and Lagrange equations that express conservation of spatial angular momentum are equivalent, but the conserved angular momentum is a complicated expression in terms of the generalized coordinates and velocities. By making the momenta conjugate to the quasivelocities primary variables, the result should be more transparent. This is anaologous to the transition to the Hamiltonian formulation from the Lagrangian formulation that is motivated by the desire to make the momenta, which are so important, primary variables.

We perform a Legendre transform from quasivelocity ω to conjugate momentum, J, and from $\hat{L}$, the Lagrangian expressed in terms of the quasivelocity, to $\hat{H}$, the Hamiltonian expressed in terms of the momentum conjugate to the quasivelocity:

$$\begin{aligned} J &= \partial_2 \hat{L}(t, q, \omega) \\ J\omega &= \hat{L}(t, q, \omega) + \hat{H}(t, q, J) \\ \omega &= \partial_2 \hat{H}(t, q, J) \\ 0 &= \partial_1 \hat{L}(t, q, \omega) + \partial_1 \hat{H}(t, q, J) \end{aligned} \tag{0.44}$$

The Poincaré equations become (on paths q and J, $\rho(t) = (t, q(t), J(t))$)

$$DJ_k = -e_k(\hat{H}) \circ \rho + \sum_{ij} (c^j_{ik} \circ q) J_j (\partial_{2,i} \hat{H} \circ \rho). \tag{0.45}$$

These are supplemented with equations that relate the generalized velocities to generalized coordinates and momenta. As a procedure

```
(define (((poincare-equations-H Hhat basis) q J) t)
  (let ((e (basis->vector-basis basis))
        (c (structure-constants basis)))
    (- ((D J) t)
       (+ (- ((e (lambda (x) (Hhat (up t x (J t))))) (q t)))
          (* (* c (J t))
             (((partial 2) Hhat) (up t (q t) (J t))))))))
```

For the free rigid body:

```
(define (Hhat A B C)
  (Lagrangian->Hamiltonian (Lhat A B C)))

(se (((poincare-equations-H (Hhat 'A 'B 'C) so3p-basis)
      Euler-path
      (down (literal-function 'Ja)
            (literal-function 'Jb)
            (literal-function 'Jc)))
     't))
```

$$\boxed{\begin{bmatrix} DJ_a(t) - \dfrac{J_c(t)\,J_b(t)}{C} + \dfrac{J_c(t)\,J_b(t)}{B} \\ DJ_b(t) + \dfrac{J_c(t)\,J_a(t)}{C} - \dfrac{J_c(t)\,J_a(t)}{A} \\ DJ_c(t) - \dfrac{J_b(t)\,J_a(t)}{B} + \dfrac{J_b(t)\,J_a(t)}{A} \end{bmatrix}}$$

These are the Euler equations written in terms of the angular momenta.

To $\tilde{L}$, the free rigid body Lagrangian written in terms of spatial angular velocities, there corresponds the Hamiltonian $\tilde{H}$, which is written in terms of the spatial components of the angular momentum. This Hamiltonian is a little too messy to show. However, the Poincaré equations show that all components of the spatial angular momentum are conserved:

```
(define (Htilde A B C)
  (Lagrangian->Hamiltonian (Ltilde A B C)))

(pe (((poincare-equations-H (Htilde 'A 'B 'C) so3-basis)
      Euler-path
      (down (literal-function 'Jx)
    (literal-function 'Jy)
    (literal-function 'Jz)))
     't))
(down ((D Jx) t) ((D Jy) t) ((D Jz) t))
```

The Poincaré equations for rotations can be written in an interesting form. For the $so(3)$ body basis:

$$DJ' = -e(\hat{H}) \circ \rho + J' \times \nabla_{J'} \hat{H} \circ \rho. \tag{0.46}$$

In this case J' is a tuple of angular momentum components on the principal axes. And for the $so(3)$ basis

$$DJ = -e(\tilde{H}) \circ \rho - J \times \nabla_J \tilde{H} \circ \rho. \tag{0.47}$$

Here J is a tuple of spatial components of the angular momentum. The commutators essentially construct cross products. Of course we have to squint and pretend that J, which is a down tuple, is a vector, to apply the vector cross product. Applying the $so(3)$ body version to $\hat{H}$ for a free rigid body, we rederive Euler's equations, written in terms of angular momenta.

Notice that since $\hat{H}$ for a free rigid body does not depend on the coordinates, we see in this form of the Poincaré equations that both $\hat{H}$ and the magnitude of J' are conserved.