Poincaré Equations

Jules Henri Poincaré (1854-1912)

Poincaré equations

- Generalize Lagrange equations
- Especially useful when the system has continuous symmetries
- Euler equations and Lagrange equations are special cases

Vector Fields

- v(f)(m) is the directional derivative of f in the direction specified by v at m.
- Let χ be a coordinate function, $q = \chi(\mathrm{m})$, then

$$\begin{aligned}
\mathrm{v}(\mathrm{f})(\mathrm{m}) &= \sum_i \partial_i(\mathrm{f} \circ \chi^{-1})(\chi(\mathrm{m}))\, b^i(\chi(\mathrm{m})) \\
&= \sum_i \mathrm{X}_i(\mathrm{f})(\mathrm{m})\, b^i(\chi(\mathrm{m})) \\
&= \sum_i \frac{\partial}{\partial \mathrm{x}^i}(\mathrm{f})(\mathrm{m})\, b^i(\chi(\mathrm{m}))
\end{aligned}$$

- $\partial/\partial \mathrm{x}^i$ are coordinate basis vector fields

Basis Fields

A vector field can be written as a linear combination of linearly independent vector fields, called a basis. For *n* dimensional manifolds, there are *n* of them.

$$\mathsf{v}(\mathsf{f})(\mathsf{m}) = \sum_i \mathsf{e}_i(\mathsf{f})(\mathsf{m})\mathsf{b}^i(\mathsf{m}).$$

A coordinate basis is a basis.

Commutators

- The commutator of two vector vields is

$$[\mathsf{v}, \mathsf{w}](\mathsf{f}) = \mathsf{v}(\mathsf{w}(\mathsf{f})) - \mathsf{w}(\mathsf{v}(\mathsf{f}))$$

- The commutator of coordinate basis fields is zero:

$$\begin{aligned}
[\mathsf{X}_i, \mathsf{X}_j](\mathsf{f})(\mathsf{m}) &= \mathsf{X}_i(\mathsf{X}_j(\mathsf{f}))(\mathsf{m}) - \mathsf{X}_j(\mathsf{X}_i(\mathsf{f}))(\mathsf{m}) \\
&= \partial_i \partial_j (\mathsf{f} \circ \chi^{-1})(\chi(\mathsf{m})) - \partial_j \partial_i (\mathsf{f} \circ \chi^{-1})(\chi(\mathsf{m})) \\
&= 0
\end{aligned}$$

Commutators (continued)

- The commutator of two vector vields is a vector field:

$$\begin{aligned} u(f) &= \sum_i X_i(f)\, b^i \\ v(f) &= \sum_i X_i(f)\, c^i \\ [u,v]\,(f) &= \sum_i X_i(f)\, a^i, \end{aligned}$$

where

$$a^i = \sum_j \left(X_j(c^i) b^j - X_j(b^i) c^j \right).$$

Structure constants

Let $\{e_i\}$ be a set of basis vector fields. The mutual commutators of each are vector fields that can be expressed in terms of the basis fields themselves:

$$[e_i, e_j]\,(f) = \sum_k c_{ij}^k e_k(f)$$

The c_{ij}^k are the structure constants of the basis.

Example

Let's find the basis on the manifold of rotations that corresponds to rotations about the spatial rectangular coordinate axes...

Use Euler angles as local coordinates:

$$R(\theta, \phi, \psi) = R_z(\phi) R_x(\theta) R_z(\psi).$$

How do we change (θ, ϕ, ψ) to add a small additional rotation about the x axis:

$$R_x(\epsilon) R_z(\phi) R_x(\theta) R_z(\psi) = R_z(\phi + a\epsilon) R_x(\theta + b\epsilon) R_z(\psi + c\epsilon).$$

Can be solved by taking derivative w.r.t. ϵ.

Example (continued)

We find:

$$\begin{aligned} e_x &= a\frac{\partial}{\partial\phi} + b\frac{\partial}{\partial\theta} + c\frac{\partial}{\partial\psi} \\ &= -\frac{\sin\phi\cos\theta}{\sin\theta}\frac{\partial}{\partial\phi} + \cos\phi\frac{\partial}{\partial\theta} + \frac{\sin\phi}{\sin\theta}\frac{\partial}{\partial\psi} \end{aligned}$$

Similarly,

$$\begin{aligned} e_y &= \frac{\cos\phi\cos\theta}{\sin\theta}\frac{\partial}{\partial\phi} + \sin\phi\frac{\partial}{\partial\theta} - \frac{\cos\phi}{\sin\theta}\frac{\partial}{\partial\psi} \\ e_z &= \frac{\partial}{\partial\phi} \end{aligned}$$

Example (continued)

The commutators satisfy:

$$\begin{aligned} [\mathrm{e}_x, \mathrm{e}_y] &= -\mathrm{e}_z \\ [\mathrm{e}_y, \mathrm{e}_z] &= -\mathrm{e}_x \\ [\mathrm{e}_z, \mathrm{e}_x] &= -\mathrm{e}_y \end{aligned}$$

Quasivelocities

Consider a function $h(q)$. On a path q, define $\bar{h}(t) = h(q(t))$. The derivative

$$D\bar{h}(t) = Dh(q(t))Dq(t)$$

is a directional derivative in the direction of the velocity $Dq(t)$.

We can also write this

$$\begin{aligned} D\bar{h}(t) &= \sum_i \frac{\partial}{\partial q^i}(h)(q(t))Dq^i(t) \\ &= \sum_i \mathsf{e}_i(h)(q(t))\omega^i(t, q(t), Dq(t)), \end{aligned}$$

for coordinate basis elements $\partial/\partial q^i$ and general basis e_i.

The ω^i are quasivelocities, linear in the generalized velocities.

Quasivelocities (continued)

For the basis corresponding to rotations about the rectangular spatial axes, the quasivelocities are the spatial angular velocities!

The components of the angular velocity on the principal axes are also quasivelocities, with respect to a basis that corresponds to rotations about the principal axes.

Derivation of the Poincaré Equations

We will derive the Poincaré equations from the action principle.

Recall that in the derivation of the Lagrange equations we encountered $\delta(Dq)$, which we were able to write as $D(\delta q)$, because we knew that Dq was the derivative of q.

In the derivation of the Poincaré equations we encounter $\delta\omega$, but now ω is not the derivative of anything, so its variation requires special attention.

Derivation (continued)

Recall $\delta_\eta f\,[q] = Dg(0)$ with $g(\epsilon) = f\,[q + \epsilon\eta]$.

Here:

$$\begin{aligned} \tilde{q}(t,\epsilon) &= q(t) + \epsilon\eta(t) \\ \tilde{f}(t,\epsilon) &= f\,[\tilde{q}(t,\epsilon)] \end{aligned}$$

then

$$\delta_\eta f\,[q]\,(t) = \partial_1 \tilde{f}(t,0)$$

Derivation (continued)

Consider $f[q](t) = h(q(t))$:

$$\begin{aligned}
\partial_0 \tilde{f}(t,\epsilon) &= \sum_i \mathsf{e}_i(h)(\tilde{q}(t,\epsilon))\omega^i(t,\tilde{q}(t,\epsilon),\partial_0\tilde{q}(t,\epsilon)) \\
&= \sum_i \mathsf{e}_i(h)(\tilde{q}(t,\epsilon))\tilde{\omega}^i(t,\epsilon) \\
\partial_1 \tilde{f}(t,\epsilon) &= \sum_i \mathsf{e}_i(h)(\tilde{q}(t,\epsilon))w^i(t,\tilde{q}(t,\epsilon),\partial_1\tilde{q}(t,\epsilon)) \\
&= \sum_i \mathsf{e}_i(h)(\tilde{q}(t,\epsilon))\tilde{w}^i(t,\epsilon).
\end{aligned}$$

Derivation (continued)

The cross partials are

$$
\begin{aligned}
\partial_1\partial_0\tilde{f} &= \sum_{i,j}(\mathrm{e}_j(\mathrm{e}_i(h))\circ\tilde{q})\tilde{\omega}^i\tilde{w}^j + \sum_i(\mathrm{e}_i(h)\circ\tilde{q})\partial_1\tilde{\omega}^i \\
\partial_0\partial_1\tilde{f} &= \sum_{j,i}(\mathrm{e}_i(\mathrm{e}_j(h))\circ\tilde{q})\tilde{w}^j\tilde{\omega}^i + \sum_j(\mathrm{e}_j(h)\circ\tilde{q})\partial_0\tilde{w}^j.
\end{aligned}
$$

The cross partials are equal, so

$$
\sum_{i,j}([\mathrm{e}_i,\mathrm{e}_j](h)\circ\tilde{q})\tilde{\omega}^i\tilde{w}^j = \sum_k(\mathrm{e}_k(h)\circ\tilde{q})\left(\partial_1\tilde{\omega}^k - \partial_0\tilde{w}^k\right).
$$

Derivation (continued)

Use the structure constants c_{ij}^k, and independence of e_i to deduce:

$$\partial_1 \tilde{\omega}^k - \partial_0 \tilde{w}^k = \sum_{i,j} \tilde{\omega}^i \tilde{w}^j c_{ij}^k \circ \tilde{q}$$

Compare

$$\delta(Dq) - D(\delta q) = 0.$$

Derivation of Poincaré equations

Define the quasi-Lagrangian $\hat{L}$

$$\hat{L}(t, q(t), \omega(t, q(t), Dq(t))) = L(t, q(t), Dq(t)).$$

The action is

$$S[q](t_0, t_1) = \int_{t_0}^{t_1} \hat{L}(t, q(t), \omega(t, q(t), Dq(t)))dt.$$

To compute the variation replace $q(t)$ with $\tilde{q}(t, \epsilon)$, take derivative w.r.t. ϵ, and set $\epsilon = 0$.

The details are very similar to the derivation of the Lagrange equations, but with an extra term involving the commutator.

The Poincaré equations

Let $\sigma(t) = (t, q(t), \omega(t))$ be a quasi-state path.

The Poincaré equations are

$$D(\partial_{2,k}\hat{L} \circ \sigma) = \mathrm{e}_k(\hat{L}) \circ \sigma + \sum_{i,j} \left(c^j_{ik} \circ q\right) \left(\partial_{2,j}\hat{L} \circ \sigma\right) \omega^i.$$

structure-constants

```
(define ((structure-constants basis) m)
  (s:map (lambda (e~j)
    (s:map (lambda (e_i)
      (s:map (lambda (e_k)
        ((e~j (commutator e_i e_k)) m))
       (basis->vector-basis basis)))
     (basis->vector-basis basis)))
   (basis->1form-basis basis)))
```

Poincare-equations

```
(define (((Poincare-equations Lhat coordsys basis) q w) t)
  (let ((e (basis->vector-basis basis))
        (c (structure-constants basis))
        (m ((point coordsys) (q t))))
    (- ((D (lambda (t)
             (((partial 2) Lhat) (up t (q t) (w t)))))
        t)
       (+ ((e (lambda (m)
                (let ((q ((chart coordsys) m)))
                  (Lhat (up t q (w t)))))) m)
          (* (* (c m)
                (((partial 2) Lhat) (up t (q t) (w t))))
             (w t))))))
```

Free Rigid Body

For a free rigid body the quasi-Lagrangian is

```
(define ((Lhat A B C) s)
  (let ((omega (ref s 2)))
    (* 1/2 (+ (* A (square (ref omega 0)))
              (* B (square (ref omega 1)))
              (* C (square (ref omega 2)))))))
```

Free Rigid Body (continued)

```
(show-expression
 (((poincare-equations (Lhat 'A 'B 'C)
                       Euler-angles so3p-basis)
   (up (literal-function 'theta)
       (literal-function 'phi)
       (literal-function 'psi))
   (up (literal-function 'omega^a)
       (literal-function 'omega^b)
       (literal-function 'omega^c)))
  't))
```

$$\begin{bmatrix} -B\omega^b(t)\,\omega^c(t) + C\omega^b(t)\,\omega^c(t) + AD\omega^a(t) \\ \\ A\omega^a(t)\,\omega^c(t) - C\omega^a(t)\,\omega^c(t) + BD\omega^b(t) \\ \\ -A\omega^a(t)\,\omega^b(t) + B\omega^a(t)\,\omega^b(t) + CD\omega^c(t) \end{bmatrix}$$

Free Rigid Body (continued)

```
(- (((Lagrange-equations (Lrigid 'A 'B 'C))
      Euler-path)
     't)
   (((Poincare-equations (Lrigid 'A 'B 'C)
                          Euler-angles Euler-basis)
      Euler-path (D Euler-path))
     't))
```

(down 0 0 0)