

Stationary Action \Rightarrow Lagrange Equations

Let q be a realizable path from $(t_1, q(t_1))$ to $(t_2, q(t_2))$.

Consider nearby paths $q + \epsilon\eta$ where $\eta(t_1) = \eta(t_2) = \mathbf{0}$.

Let

$$\begin{aligned} f(0 + \epsilon) &= S[q + \epsilon\eta](t_1, t_2) \\ &= \int_{t_1}^{t_2} L(t, q(t) + \epsilon\eta(t), Dq(t) + \epsilon D\eta(t)) dt \\ &= \int_{t_1}^{t_2} L(t, q(t), Dq(t)) dt \\ &\quad + \epsilon \left(\frac{d}{d\delta} \int_{t_1}^{t_2} L(t, q(t) + \delta\eta(t), Dq(t) + \delta D\eta(t)) dt \right) \Big|_{\delta=0} \\ &\quad + \dots \\ &= f(0) + \epsilon Df(0) + \dots \end{aligned}$$

So

$$Df(0) = \frac{d}{d\delta} \int_{t_1}^{t_2} L(t, q(t) + \delta\eta(t), Dq(t) + \delta D\eta(t)) dt.$$

$$\begin{aligned}
Df(0) &= \left. \left(\frac{d}{d\delta} \int_{t_1}^{t_2} L(t, q(t) + \delta\eta(t), Dq(t) + \delta D\eta(t)) dt \right) \right|_{\delta=0} \\
&= \int_{t_1}^{t_2} \{\partial_1 L(t, q(t), Dq(t))\eta(t) + \partial_2 L(t, q(t), Dq(t))D\eta(t)\} dt \\
&= \int_{t_1}^{t_2} \partial_1 L(t, q(t), Dq(t))\eta(t) dt \\
&\quad + \partial_2 L(t, q(t), Dq(t))\eta(t)|_{t_1}^{t_2} \\
&\quad - \int_{t_1}^{t_2} \frac{d}{dt}(\partial_1 L(t, q(t), Dq(t)))\eta(t) dt.
\end{aligned}$$

Now

$$\partial_2 L(t, q(t), Dq(t))\eta(t)|_{t_1}^{t_2} = 0.$$

So

$$Df(0) = \int_{t_1}^{t_2} \{\partial_1 L \circ \Gamma[q] - D(\partial_2 L \circ \gamma[q])\}\eta.$$

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But, for stationary action $Df(0) = 0$.

And because η is arbitrary except for $\eta(t_1) = \eta(t_2) = \mathbf{0}$ we obtain

$$D(\partial_2 L \circ \Gamma[q]) - \partial_1 L \circ \Gamma[q] = \mathbf{0},$$

the Euler-Lagrange Equations.