6.837 Linear Algebra Review

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Overview

• Basic matrix operations (+, -, *)
• Cross and dot products
• Determinants and inverses
• Homogeneous coordinates
• Orthonormal basis
Additional Resources

• 18.06 Text Book
• 6.837 Text Book
• 6.837-staff@graphics.lcs.mit.edu
• Check the course website for a copy of these notes
What is a Matrix?

- A matrix is a set of elements, organized into rows and columns.

\[
\begin{bmatrix}
  a & b \\
  c & d \\
\end{bmatrix}
\]
Basic Operations

- Addition, Subtraction, Multiplication

\[
\begin{bmatrix}
a & b \\
c & d
\end{bmatrix} + \begin{bmatrix}
e & f \\
g & h
\end{bmatrix} = \begin{bmatrix}
a + e & b + f \\
c + g & d + h
\end{bmatrix}
\]

Just add elements

\[
\begin{bmatrix}
a & b \\
c & d
\end{bmatrix} - \begin{bmatrix}
e & f \\
g & h
\end{bmatrix} = \begin{bmatrix}
a - e & b - f \\
c - g & d - h
\end{bmatrix}
\]

Just subtract elements

\[
\begin{bmatrix}
a & b \\
c & d
\end{bmatrix} \begin{bmatrix}
e & f \\
g & h
\end{bmatrix} = \begin{bmatrix}
ae + bg & af + bh \\
ce + dg & cf + dh
\end{bmatrix}
\]

Multiply each row by each column
• Is $AB = BA$? Maybe, but maybe not!

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} ae + bg & \cdots \\ \cdots & \cdots \end{bmatrix} \begin{bmatrix} e & f \\ g & h \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} ea + fc & \cdots \\ \cdots & \cdots \end{bmatrix}$$

• Heads up: multiplication is NOT commutative!
Vector Operations

- **Vector**: 1 x N matrix
- Interpretation: a line in N dimensional space
- Dot Product, Cross Product, and Magnitude defined on vectors only

\[ \vec{v} = \begin{bmatrix} a \\ b \\ c \end{bmatrix} \]
Vector Interpretation

- Think of a vector as a line in 2D or 3D
- Think of a matrix as a transformation on a line or set of lines

\[
\begin{bmatrix}
  x \\
  y
\end{bmatrix}
\begin{bmatrix}
  a & b \\
  c & d
\end{bmatrix}
= 
\begin{bmatrix}
  x' \\
  y'
\end{bmatrix}
\]
Vectors: Dot Product

- Interpretation: the dot product measures to what degree two vectors are aligned

\[ \mathbf{A} + \mathbf{B} = \mathbf{C} \]
(use the head-to-tail method to combine vectors)
Think of the dot product as a matrix multiplication

\[ a \cdot b = ab^T = \begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix} = ad + be + cf \]

The magnitude is the dot product of a vector with itself

\[ \|a\|^2 = aa^T = \sqrt{aa + bb + cc} \]

The dot product is also related to the angle between the two vectors – but it doesn’t tell us the angle

\[ a \cdot b = \|a\|\|b\|\cos(\theta) \]
Vectors: Cross Product

• The cross product of vectors A and B is a vector C which is perpendicular to A and B
• The magnitude of C is proportional to the cosine of the angle between A and B
• The direction of C follows the right hand rule – this why we call it a “right-handed coordinate system”

\[ ||a \times b|| = ||a|| ||b|| \sin(\theta) \]
Inverse of a Matrix

- Identity matrix:
  \( AI = A \)

- Some matrices have an inverse, such that:
  \( AA^{-1} = I \)

- Inversion is tricky:
  \( (ABC)^{-1} = C^{-1}B^{-1}A^{-1} \)

  Derived from non-commutativity property

\[
I = \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}
\]
Use for inversion
If $\text{det}(A) = 0$, then $A$ has no inverse
Can be found using factorials, pivots, and cofactors!
Lots of interpretations – for more info, take 18.06

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$\text{det}(A) = ad - bc$$

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$
Determinant of a Matrix

\[
\begin{vmatrix}
  a & b & c \\
  d & e & f \\
  g & h & i
\end{vmatrix}
= aei + bfg + cdh - afh - bdi - ceg
\]

Sum from left to right
Subtract from right to left
Note: N! terms
Inverse of a Matrix

1. Append the identity matrix to A
2. Subtract multiples of the other rows from the first row to reduce the diagonal element to 1
3. Transform the identity matrix as you go
4. When the original matrix is the identity, the identity has become the inverse!
Homogeneous Matrices

• Problem: how to include translations in transformations (and do perspective transforms)
• Solution: add an extra dimension

\[
\begin{bmatrix}
1 & 1 \\
1 & 1 \\
1 & 1 \\
\end{bmatrix}
\begin{bmatrix}
x \\
y \\
z \\
1
\end{bmatrix}
= 
\begin{bmatrix}
x \\
y \\
z \\
1
\end{bmatrix}
\]
Orthonormal Basis

- Basis: a space is totally defined by a set of vectors – any point is a linear combination of the basis
- Ortho-Normal: orthogonal + normal
- Orthogonal: dot product is zero
- Normal: magnitude is one
- Example: X, Y, Z (but don’t have to be!)
Orthonormal Basis

\[ x = [1 \ 0 \ 0]^T \quad x \cdot y = 0 \]
\[ y = [0 \ 1 \ 0]^T \quad x \cdot z = 0 \]
\[ z = [0 \ 0 \ 1]^T \quad y \cdot z = 0 \]

\textbf{X, Y, Z} is an orthonormal basis. We can describe any 3D point as a linear combination of these vectors.

How do we express any point as a combination of a new basis \textbf{U, V, N}, given \textbf{X, Y, Z}?
Orthonormal Basis

\[
\begin{bmatrix}
a & 0 & 0 \\
0 & b & 0 \\
0 & 0 & c \\
\end{bmatrix}
\begin{bmatrix}
u_1 & v_1 & n_1 \\
u_2 & v_2 & n_2 \\
u_3 & v_3 & n_3 \\
\end{bmatrix}
= \begin{bmatrix}
a \cdot u + b \cdot u + c \cdot u \\
a \cdot v + b \cdot v + c \cdot v \\
a \cdot n + b \cdot n + c \cdot n \\
\end{bmatrix}
\]

(not an actual formula – just a way of thinking about it)

To change a point from one coordinate system to another, compute the dot product of each coordinate row with each of the basis vectors.