

## Triangles are Minimal:

Triangles are determined by 3 points or 3 edges We can define a triangle in terms of three points, for example:
$\left(x_{1}, y_{1}\right),\left(x_{2}, y_{2}\right),\left(x_{3}, y_{3}\right)$

Or we can define a triangle interm of its three edges, for example:

$$
A_{1} x+B_{1} y+C_{1}=0, \quad A_{2} x+B_{2} y+C_{2}=0, \quad A_{3} x+B_{3} y+C_{3}=0
$$

Why does it seem to take more parameters to represent the edges than the points? (Hint: Edge equations are Homogeneous)

As a result, triangles are mathematically very simple. The math involved in scan converting triangles involves only simple linear equations.

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## Triangles are Always Convex Polygons

What does it mean to be a convex?


An object is convex if and only if any line segment connecting two points on its boundary is contained entirely within the object or one of its boundries.

Why is being convex important?
Because no matter how a triangle is oriented on the screen a given scan line will contain only a single segment or span of that triangle.

## Fractional Offsets



We can use ceiling to find the leftmost pixel in span and floor to find the rightmost pixel.
The trick comes when interpolating color values. It is straightforward to interpolate along the edges, but you must be careful when offsetting from the edge to the pixel's center.

That's all that I'm going to give away on this subject. You'll need to work out the details for yourself in you Project \#2.

## Rasterizing Triangles using Edge Equations

An edge equation is simply a discriminating function like those we used in our curve and line-drawing algorithms.

An edge equation segments a planar region into three parts, a boundary, and two half-spaces. The boundary is identified by points where the edge equation is equal to points where the edge equation is equal to zero. The half-spaces are distiguished by ifferences in the edge equation's sign. We can choose which half-space gives a positive sign by multiplication by -1


## Notes on using Edge Equations

- Compute edge equations from vertices
- Orient edge equations
- Compute a bounding box
- Scan through pixels in bounding box evaluating the edge equations
- When all three are positive then draw the pixel.



## Example Implementation

First we define a few useful objects

$$
\begin{array}{cl}
\text { public class Vertex2D } i & \\
\text { public float } x, y ; & \text { // coordinate of vertex } \\
\text { public int argb; } & \text { // color of vertex }
\end{array}
$$

public vertex2D() i
public Vertex2D(float xval, float yval, int cval)
$\mathrm{x}=\mathrm{xval} ;$
$\mathrm{y}=\mathrm{yval} ;$
$\mathrm{y}=\mathrm{yval} ;$
$\mathrm{argb}=$ cval
, 1
A Drawable interface
import Raster;
public abstract interface Drawable
public abstract void Draw (Raster r)
The edge equations use integer coefficients for speed. This implementation uses 12 fractional bits. (Not my first version)

## Edge Equation Coefficients

The edge equation coefficients are computed using the coordinates of the two vertices. Each point determines an equation in terms of our three unknowns, $A, B$, and $C$.

$$
\begin{aligned}
& A x_{0}+B y_{0}+C=0 \\
& A x_{1}+B y_{1}+C=0
\end{aligned}
$$

We can solve for A and B in terms of C by setting up the following linear system.

$$
\left[\begin{array}{ll}
x_{0} & y_{0} \\
x_{1} & y_{1}
\end{array}\right]\left[\begin{array}{l}
A \\
B
\end{array}\right]=-C\left[\begin{array}{l}
1 \\
1
\end{array}\right]
$$

Multiplying both sides by the matrix inverse.

$$
\left[\begin{array}{l}
A \\
B
\end{array}\right]=\frac{-C}{x_{0} y_{1}-x_{1} y_{0}}\left[\begin{array}{cc}
y_{1} & -y_{0} \\
-x_{1} & x_{0}
\end{array}\right]\left[\begin{array}{l}
1 \\
1
\end{array}\right]
$$

If we choose $C=x_{0} y_{1}-x_{1} y_{0}$, then we get $A=y_{0}-y_{1}$ and $B=x_{1}-x_{0}$.
Why could we just choose $C$ ?

## Numerical Precision of the Coefficient

Computers represent floating-point number internally in a format similar to scientific notation. The very worse thing that you can do with numbers represented in scientific notation is subtract numbers of similar magnitude. Here is what happens:

$$
\underline{1.234} \times 10^{2}-\underline{1.233} \times 10^{2}=\underline{1.000} \times 10^{0}
$$

We loose most of the significant digits in our result.
In the case of triangles, these sort of precision problems to occur frequently, because in general the vertices of a triangle are close to each other.

$$
x_{0} \approx x_{1} \text { and } y_{0} \approx y_{1} \text { thus } x_{0} y_{1}-x_{1} y_{0} \approx 0
$$

Thankfully, we can avoid this subtraction of large numbers when computing an expression for $C$. Given that we know $A$ and $B$ we can solve for $C$ as follows

$$
C_{0}=-A x_{0}-B y_{0} \text { or } C_{1}=-A x_{1}-B y_{1}
$$

To eliminate any bias toward either vertex we will average of these C values

$$
C_{\text {ave }}=\frac{-\left(A\left(x_{0}+x_{1}\right)+B\left(y_{0}+y_{1}\right)\right)}{2}
$$

## EdgeEquation Object

class EdgeEqn \{
public final static int FRACBITS $=12$
public int A, B, C;
public int flag;
public EdgeEqn (Vertex2D v0, Vertex2D v1) i
double $a=v 0 \cdot \mathrm{y}-\mathrm{v} 1 \cdot \mathrm{y}$;
double $\mathrm{b}=\mathrm{v} 1 \cdot \mathrm{x}-\mathrm{vo}$
double $c=-0.5 f^{*}(a *(v 0 . x+v 1 . x)+b *(v 0 . y+v 1 . y))$;
$\mathrm{A}=$ (int) $(\mathrm{a} *$ ( $1 \ll \mathrm{FRACBITS}$ ) )
$\mathrm{B}=$ (int) $(\mathrm{b} *$ ( $1 \ll$ FRACBITSS) $)$
$c=$ (int) ${ }^{\text {flag }}=0$;
if ( $A>=0$ ) $\begin{aligned} & \text { flag }+=8 ; \\ & \text { if }(B>=0) \\ & \text { flag }+=1 ; ~\end{aligned}, ~$
\}
public void flip() (
$\mathrm{A}=-\mathrm{A} ;$
$\mathrm{B}=-\mathrm{B} ;$
$\mathrm{B}=-\mathrm{B} ;$
$\mathrm{c}=-\mathrm{C} ;$
\}
public int evaluate (int $x$, int $y$ ) ( return $\left(A * \mathbf{x}+\mathrm{B}_{\mathrm{y}} \mathrm{y}+\mathrm{C}\right)$;
\}


## Triangle Object

```
public class FlatTri imp
```


protected Edgeigqn
protected
int
coll
protected int color;
protected int area;

public FlatTri() \& // for future extension

$\mathrm{v}=$ new Vertex2D[3]
$\mathrm{v}[01=\mathrm{vo} ;$
$\mathrm{v}[1]=\mathrm{v} 1$;
$\mathrm{v}[2]=\mathrm{v} 2$;



$\mathrm{a}=(\mathrm{a}+\mathrm{a}+3) / \mathbf{c}_{6}$
$\mathrm{r}=(\mathrm{r}+\mathrm{r}+3)$
b
$\mathrm{a}=(\mathrm{r}+\mathrm{r}+3) / \begin{gathered}6 ; \\ \mathrm{r}=(\mathrm{c}+\mathrm{g}+3) \\ \mathrm{g}=(\mathrm{b} ; \\ \mathrm{b}=(\mathrm{b}+\mathrm{b}+3) \\ 6 ;\end{gathered}$
color $=(a \ll 24)|(x \ll 16)|(g \ll 8) \mid b ;$

## Lecture 6 .- 6.837 Fall 0

## The Draw Method

 int widt $=$ r.getwidth ();
int height $=\tau$ geteineight $)$
in
f (!trianglesetup (width, height)) return)


$y_{\text {Min }}^{*} *=$ width;
yMax $*=$ width;

$\qquad$

beennside $=$ true:


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Lecture 6
Slide 16

## Explain Two Speed Hacks

Most everything in our Draw( ) method is straightforward, with two exceptions.

```
for (x = xMin;
    (x = xMin; x <= xMax; x++) {
            ff((e0|e1|e2)>=0) f
            cflag++;
            }else if (xflag != 0) break
            e0 +=A0;
            el += A1;
}
``` - All three edges are tested with a single comparison by oring together the three edges and checking if the result is
positive. If any one of the three is negative then its sign-bit will be set to a 1 , and the result of the or will be negative.
- Since triangles are convex, we can only be inside for a single interval on any given scanline. The xflag variable is used to keep track of when we exit the triangle's interior. If ever we find ourselves outside of the triangle having already set some pixels on the span then we can skip over the remainder of the scanline.

\section*{Triangle SetUp Method}
    if (egge \(==\) null) edges \(=\) new Eqgeeqn \(n\), 3 ;
Compute the three edge equations


    // Trick \#1: Orient edges so that the triangle's interior 1ies within all of their
// positive half-spaces. Assuring that the area is positive accomplishes this
    // positive half-spaces. Assuring that the
area \(=\) edge \([0] . c+\) edge \([11 . \mathrm{c}+\) edge \([2] . \mathrm{c} ;\)
    if (area \(=001\)
return false;
    \({ }_{\text {f }}^{\text {f return false }}\),






    \(x\) min \(=(\) int \()(v[s o r t[y f l a g] ~\)




ymax \(=\) (ymax
return true ;

\section*{Tricks Explained}

In this method we did two critical things. We orient the edge equations, and we compute the bounding box.

From analytic geometry we know that the area of a triangle \(\left\{\left(x_{1}, y_{1}\right),\left(x_{2}, y_{2}\right),\left(x_{3}, y_{3}\right)\right\}\) is:
\[
\text { Area }=\frac{1}{2} \operatorname{det}\left[\begin{array}{ccc}
x_{1} & x_{2} & x_{3} \\
y_{1} & y_{2} & y_{3} \\
1 & 1 & 1
\end{array}\right]
\]

The area is positive if the vertices are counterclockwise and negative if clockwise
An aside:
In terms of our discriminator, what does a positive \(C\) imply?

\section*{Why Positive Area Imply a Positive Interior}



1. The area of each sub-triangle gives the edge equation's sign at the origin
2. Assume a positive area
thus, the sum of the sub-triangles areas are positive
3. Each point within the triangle falls within exactly one subtriangle thus, subtriangles with negative areas will lie outside of the triangle
4. Since the negative-subtriangle areas are outside of the triangle the edge equations are positive inside
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\section*{Exercise \#2}

Examine the code for computing the bounding box of the triangle in
EdgeEqn( ) and FlatTri.triangleSetup( ).
- Explain what is being saved in the EdgeEqn.flag
- Explain the contents FlatTri.sort[ ]
- Explain how the bounding box is computed
- Discuss the advantages and disadvantages of this approach
- Write down I give up if this exercise takes you more than 1 Hr .

Limit your discussion to one single-sided sheet of 8.5 by 11 paper.
Turn this exercise in the next time that we meet.

\section*{Triangle Rasterizer Demonstration}

Press the left mouse button above to render a simple
scene with the FlatTri rasterizer
Press the left mouse button above to render a more complicated scene with the FlatTri rasterizer.

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\section*{Interpolating Parameters Within a Triangle}

Currently, our triangle scan-converter draws only solid colored triangles. Next we'll discuss how to smoothly vary parameters as we fill the triangle. In this case the parameters that are interpolated are the red, green, and blue components of the color. Later on, when we get to 3D techniques, we'll also interpolate other parameters such as the depth at each point on the triangle.
First, let's frame the problem. At each vertex of a triangle we have a parameter, say its redness. When we actually draw the vertex, the specified shade of red is exactly what we want, but at other points we'd like some sort of smooth transition between the values given. This situation is shown to the right:
Notice that the shape of our desired redness function is planar. Actually, it is a special class of plane where there exists a corresponding point for every \(x\) - \(y\) coordinate. Planes of this type can always be expressed in the following
form:
\[
z=A x+B y+C
\]
This equation should appear familiar. It has the same form as our edge equations. Given the redness of the three vertices, we can set up the following linear system.
\[
\left[\begin{array}{l}
r_{0} \\
r_{1} \\
r_{2}
\end{array}\right]=\left[\begin{array}{lll}
x_{0} & y_{0} & 1 \\
x_{1} & y_{1} & 1 \\
x_{2} & y_{2} & 1
\end{array}\right]\left[\begin{array}{l}
A_{r} \\
B_{r} \\
C_{r}
\end{array}\right]
\]
with the solution:
\[
\frac{1}{\text { 2area } a}\left[\begin{array}{ccc}
y_{1}-y_{2} & y_{2}-y_{0} & y_{0}-y_{1} \\
x_{2}-x_{1} & x_{0}-x_{2} & x_{1}-x_{0} \\
x_{1} y_{2}-x_{2} y_{1} & x_{2} y_{0}-x_{0} y_{2} & x_{0} y_{1}-x_{1} y_{0}
\end{array}\right]\left[\begin{array}{l}
r_{0} \\
r_{1} \\
r_{2}
\end{array}\right]=\left[\begin{array}{c}
A_{r} \\
B_{r} \\
C_{r}
\end{array}\right]
\]
By the way, we've already computed these matrix entries, they're exactly the coefficients of our edge equations
\[
\frac{1}{\text { 2area }}\left[\begin{array}{lll|l}
A_{2} & A_{3} & A_{1} \\
B_{2} & B_{3} & B_{1} & r_{0} \\
r_{1} \\
C_{2} & C_{3} & C_{1}
\end{array}\right]=\left[\begin{array}{l}
r_{2}
\end{array}\right]=\left[\begin{array}{l}
A_{r} \\
B_{r} \\
C_{r}
\end{array}\right]
\]
So the all the additional work that we need to do to interpolate is a single matrix multiplication and compute the equivalent of an extra edge equation for each parameter

\section*{Smooth Triangle Object}
import Raster;
import Drawable
mport Vertex2D;
import FlatTri;
public class SmoothTri extends Flattri implements Drawable boolean 1sFlat
public SmoothTri (Vertex2D v0, Vertex2D v1, Vertex2D v2) \(\checkmark[0]=\mathrm{vo}\);
\(\mathrm{v}[0]=\mathrm{v} 0\)
\(\mathrm{v}[1]\)
v 1
\(\mathrm{v}[2]=\mathrm{v} 2\);
/*
/ check if all vertices are the same color
*/ \({ }_{\text {isflat }}=(\mathrm{vo} . \mathrm{argb}==\mathrm{v} 1 . \mathrm{argb}) \& \&(\mathrm{v} 0 . \mathrm{argb}==\mathrm{v} 2 . \mathrm{argb})\)
if (isFlat) \(\{\)
else \(1=\) vo.argb
color \(=0\);
\}
Scale is always non zero and positive. This zero value indicates that it has not been computed yet *scale \(=-1\);


\section*{Modified Draw( ) Method}

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```

\square

We've added two new instance variables. The first is simply an optimization that detects the case when all three vertices are the same color. In this case we'll call the slightly faster FlatTri methods that we inherited. The second is a scale factor that we'll disscuss next.

Next we add a new method to compute the plane equations of our parameters. The
PlaneEqn() method performs the required matrix multiply and avoids computing the inverse of the triangle area more than once.

```
Mic void planeEqn(
    int Ap, Bp, Cp;
    *)
    double spo = scale * po;
    *)
```



```
*)
eqn[0] =Ap;
```





## Smooth Triangle Results

Press the left mouse button above to render a simple scene with the SmoothTri rasterizer

Press the left mouse button above to render a more Today's Code: Vertex2D.java, Drawable,java, FlatTri.java, SmoothTri.java, and Triangles.java

## A Post-Triangle World

Are triangles really the best rendering primitive?
$100,000,000$ primitive models displayed on 2,000,000 pixel displays.

Even even if we assume that only $10 \%$ of the primities are visible, and they are uniformly distributed over the whole screen, thats still 5
primitives/pixel. Remember, that in order to draw a single triangle we must specify 3 vertices, determine three colors, and interpolte within 3 edges. On average, these triangle will impact pnly a fraction of a pixel.



