What’s Next?

Step up one dimension from curves to surfaces.

- Theoretical definition
- Discrete representations
- Higher dimensionality
Easier transition.

Not entirely true:
e.g. topology of 3-manifolds
Our Focus

$\subseteq \mathbb{R}^3$

Embedded geometry

http://web.mit.edu/manoli/crust/www/slides/piggy.jpg
What is an embedded surface?
Warm Up: Parametric Surface
Pathological Cases

\[ f(u, v) = (u, u^2, \cos u) \]

\[ f(u, v) = (0, 0, 0) \]

\[ f(u, v) = (u, v^3, v^2) \]

What condition do we need to add?
Review: Jacobian Matrix

\( f : \mathbb{R}^m \rightarrow \mathbb{R}^n \)

Jacobian matrix:

\[
(Df)^i_j = \left( \frac{\partial f^i}{\partial x^j} \right)
\]
Regularity (Injectivity/One-to-One) Condition

\[ f : \mathbb{R}^m \rightarrow \mathbb{R}^n \]

Matrix condition:

\[ Df \text{ full rank} \]
Moving Away from Parametric Surfaces

One function isn’t enough!

Major difference from curves!

https://en.wikipedia.org/wiki/Triple_torus
Recall:
Differential Geometry Definition

\[ \gamma_p : (a, b) \rightarrow C \cap U \]
A surface is a set of points with certain properties.

It is not a function.
Theoretical Definition of Surface

\[ \subseteq \mathbb{R}^3 \]

\[ \subseteq \mathbb{R}^2 \]
**Theoretical Definition: (Sub)Manifold**

**Definition** (Submanifold of \( \mathbb{R}^n \), with and without boundary). A set \( \mathcal{M} \subseteq \mathbb{R}^n \) is an \( m \)-dimensional submanifold of \( \mathbb{R}^n \) if for each \( p \in \mathcal{M} \) there exist open sets \( U \subseteq \mathbb{R}^m \), \( W \subseteq \mathbb{R}^n \) and a function \( g : U \cap \mathcal{H}_m \rightarrow \mathcal{M} \cap W \) such that \( p \in W \) and \( g \) is a one-to-one and smooth map whose Jacobian is rank-\( m \) and admitting a continuous inverse \( g^{-1} : W \cap \mathcal{M} \rightarrow U \).

\[
\mathcal{H}_m := \{ x \in \mathbb{R}^m : x^m \geq 0 \}
\]
Differential Geometer’s Mantra

A surface is locally planar.
Tangent Space

\[ T_p \mathcal{M} = \gamma'(0), \text{ where } \gamma(0) = p \]
Recall: Differential

\[ df_{x_0}(v) := \lim_{h \to 0} \frac{f(x_0 + hv) - f(x_0)}{h} \]

**Proposition.** \( df_{x_0} \) is a linear operator.

\[ df_{x_0}(v) = Df(x_0) \cdot v \]

Note: Technically we derived the 1D version. Nothing changes!
Tangent Space

\[ T_p M = \gamma'(0), \text{ where } \gamma(0) = p = \text{image}(dg_{g^{-1}(p)}) \]

\[ M \subseteq \mathbb{R}^n \]

\[ U \subseteq \mathbb{R}^m \]
Normal Space

\[ N_p\mathcal{M} := (T_p\mathcal{M})^\perp \]

\[ \mathcal{M} \subseteq \mathbb{R}^n \]

\[ U \subseteq \mathbb{R}^m \]
Orientable Submanifold

Admits a continuous map

\[ n(p) : M \setminus \partial M \rightarrow S^{n-1} \]

with

\[ n(p) \in N_pM \]

Orientable

Not Orientable

Figure 3. Non-orientable surface with second order saddles.
(all photos, Phillip Geller)

Brend Collins (Gower, Missouri) is a sculptor who has presented his work at AM93, AM95, and AM97. His recent collaboration with Carlo Séquin has resulted in new forms that represent the leading edge of sculpture influenced by mathematics.
More General Definition: Manifold

**Definition 4.2 (Manifold).** An $m$-dimensional (topological) manifold $\mathcal{M}$ is a Hausdorff space for which each $p \in \mathcal{M}$ admits open sets $U \subseteq \mathbb{R}^m, W \subseteq \mathcal{M}$ and a homeomorphism (continuous map with continuous inverse) $g : U \rightarrow W$.

**To think about:**
No notion of normal!
Tangent vectors exist but have no length!
How do you detect orientability?

http://www.math.sjsu.edu/~simic/Pics/Calabi-Yau.jpg
What is a discrete surface?
How do you store it?
Common Representation

Triangle mesh

http://graphics.stanford.edu/data/3Dscanrep/stanford-bunny-cebal-ssh.jpg
http://www.stat.washington.edu/wxs/images/BUNMID.gif
Triangle Mesh

\[ V = (v_1, v_2, \ldots, v_n) \subseteq \mathbb{R}^3 \]

\[ E = (e_1, e_2, \ldots, e_k) \subseteq V \times V \]

\[ F = (f_1, f_2, \ldots, f_m) \subseteq V \times V \times V \]

Plus manifold topological conditions
Dimensionality Structure

- **Vertex**: Dimension 0
- **Edge**: Dimension 1
- **Face**: Dimension 2

Simplicial complex
Is This a Discrete Surface?
Topology [\textit{tuh-pol-uh-jee}]:
The study of geometric properties that remain invariant under certain transformations

http://dictionary.reference.com/browse/topology?s=t
Mesh Topology vs. Geometry

Geometry: “This vertex is at \((x,y,z)\).”
Mesh Topology vs. Geometry

Topology:
“These vertices are connected.”
To read: More general story

“Orientable combinatorial manifold”
Nonmanifold Edge
1. Each *edge* is incident to one or two faces

2. *Faces* incident to a vertex form a closed or open fan

Manifold Triangle Mesh

1. Each \textbf{edge} is incident to one or two faces

2. Faces incident to a vertex form a closed or open fan

Assume meshes are manifold (for now)
Easy-to-Violate Assumption

“Triangle soup”
Basic Observation

Piecewise linear faces are reasonable building blocks.
Additional Advantages

- Simple to render
- Arbitrary topology possible
- Basis for subdivision, refinement
Invalid Meshes vs. Bad Meshes

Nonuniform areas and angles
Why is Meshing an Issue?

How to you interpret one value per vertex?

Returning to Topology: Valence

Valence = 6

Synonym: Degree
Euler Characteristic for Meshes

\[ V - E + F := \chi \]

\[ \chi = 2 - 2g \]

\[ g = 0 \quad \text{g} = 1 \quad \text{g} = 2 \]
Consequences for Triangle Meshes

\[ V - E + F := \chi \]

“Each edge is adjacent to two faces. Each face has three edges.”

\[ 2E = 3F \]

Closed mesh: Easy estimates!
Consequences for Triangle Meshes

\[ V - \frac{1}{2} F := \chi \]

“Each edge is adjacent to two faces. Each face has three edges.”

\[ 2E = 3F \]

Closed mesh: Easy estimates!
Consequences for Triangle Meshes

\[ V - \frac{1}{2} F := \chi \]

\[ F \approx 2V \]

Big number \quad \text{Small number}

Closed mesh: Easy estimates!
Consequences for Triangle Meshes

\[ E \approx 3V \]
\[ F \approx 2V \]

average valence \( \approx 6 \)

Why?!
Orientability

http://www.cse.ohio-state.edu/~tamaldey/isotopic.html
Continuous field of normal vectors
What happens on edges/vertices?
Right-Hand Rule

http://viz.aset.psu.edu/gho/sem_notes/3d_fundamentals/html/3d_coordinates.html
http://mathinsight.org/stokes_theorem.orientation
Discrete Orientability

Normal field isn’t continuous
Data Structures for Surfaces

Must represent geometry and topology.
Simplest Format

\[
x_1 \ y_1 \ z_1 \ / \ x_2 \ y_2 \ z_2 \ / \ x_3 \ y_3 \ z_3
\]

\[
x_1 \ y_1 \ z_1 \ / \ x_2 \ y_2 \ z_2 \ / \ x_3 \ y_3 \ z_3
\]

\[
x_1 \ y_1 \ z_1 \ / \ x_2 \ y_2 \ z_2 \ / \ x_3 \ y_3 \ z_3
\]

\[
x_1 \ y_1 \ z_1 \ / \ x_2 \ y_2 \ z_2 \ / \ x_3 \ y_3 \ z_3
\]

\[
x_1 \ y_1 \ z_1 \ / \ x_2 \ y_2 \ z_2 \ / \ x_3 \ y_3 \ z_3
\]

No topology!

\texttt{glBegin(GL_TRIANGLES)}
Factor Out Vertices

Shared vertex structure

.f 1 5 3
.f 5 1 2
...
v 0.2 1.5 3.2
.v 5.2 4.1 8.9
...
.obj format
Simple Mesh Smoothing

\[
\text{for } i=1 \text{ to } n \\
\text{for each vertex } v \\
\quad v = .5*v + .5*(\text{average of neighbors});
\]
Neighboring vertices to a vertex
- Neighboring faces to an edge
- Edges adjacent to a face
- Edges adjacent to a vertex
- ...

Mostly localized
Typical Queries

- Neighboring vertices to a vertex
- Neighboring faces to an edge
- Edges adjacent to a face
- Edges adjacent to a vertex
- ...

Mostly localized
Pieces of Halfedge Data Structure

- Vertices
- Faces
- Half-edges

Structure tuned for meshes
Oriented edge

Halfedge?

Associated with single face!

Oriented edge
Halfedge Data Types

Vertex stores:
• Arbitrary outgoing halfedge
Halfedge Data Types

Face stores:
• Arbitrary adjacent halfedge
Halfedge Data Types

Halfedge stores:
- Flip
- Next
- Face
- Vertex
Iterate(v):
startEdge = v.out;
e = startEdge;
do
    process(e.flip.from)
    e = e.flip.next
while e != startEdge
Streaming Compression of Triangle Meshes

Martin Isenburg¹, Peter Lindstrom², Jack Snoeyink¹

¹ University of North Carolina at Chapel Hill ² Lawrence Livermore National Labs

EUROGRAPHICS 2011 / M. Chen and O. Deussen (Guest Editors)

SQuad: Compact Representation for Triangle Meshes

Topraj Gurung¹, Daniel Lancy², Peter Lindstrom², Jarek Rossignac¹

¹ Georgia Institute of Technology ² Lawrence Livermore National Laboratory
Scalar Functions

Map points to real numbers
Discrete Scalar Functions

Map vertices to real numbers

\[ f \in \mathbb{R}^{|V|} \]
What is the integral of $f$?

$$\int_{M} f \, dA$$
Discrete version of $dA$
Dual Complex

Valence 3
One Surface, Two Meshes
One Surface, Two Halfedges
Quad Edge
$e \rightarrow \text{Rot} \rightarrow \text{Rot} = e \rightarrow \text{Flip}$
Topological Operations

- **Vertex Removal**
- **Edge Collapse**
- **Face Collapse**
Topological Operations

Original Mesh Segment

Vertex Removal  Edge Collapse  Face Collapse

Necessary bookkeeping for each operation?
Complex data structures enable simpler traversal at cost of more bookkeeping.
Implicit surfaces

Application of Implicit Surfaces

Smoothed-particle hydrodynamics (SPH)

http://www.itsartmag.com/features/cgfluids/
https://developer.nvidia.com/content/fluid-simulation-alice-madness-returns
Polynomial/rational patches
Not the Only Geometric Representation

Subdivision Surfaces

https://imagecomputing.net/damien.rohmer/teaching/2018_2019/semester_1/m2_mpri_cg_viz/class/01_surface_representation/content/035_subdivision_surfaces/index.html
Aside

Halfedge suited for subdivision!

Catmull-Clark
Not the Only Geometric Representation

Volumetric imaging

http://www.colin-studholme.net/software/rview/rvmanual/morphtool5.gif
Surfaces from Volumes

Marching cubes: Isosurface extraction

Volumetric extraction

http://en.wikipedia.org/wiki/Marching_cubes
Not the Only Geometric Representation

Point clouds