Previously

Map between two shapes.
What happens if you compose these maps?
Q: What do you expect if you compose around a cycle?
Cycle consistency

Composing maps in a cycle yields the identity
You should have a good reason if your correspondences are inconsistent.
An Unpleasant Constraint

$$\phi_1(\phi_2(\phi_3(x))) = \text{Id}$$

Cycle consistency
Contrasting Viewpoint

Many possible pairwise matches!

Additional data should help!
Sampling of methods for consistent correspondence.

- Spanning tree
- Inconsistent cycle detection
- Convex optimization
Simultaneously optimize all maps in a collection.
Joint Matching: Simplest Formulation

- **Input**
  - $N$ shapes
  - $N^2$ maps (see last lecture)

- **Output**
  - Cycle-consistent approximation
Spanning Tree: Original Context

"Automatic Three-Dimensional Modeling from Reality" (Huber, 2002)

Multi-view registration
Unsurprisingly...

Given: Model graph $G = (S, E)$
Find: Largest consistent spanning tree

“Automatic Three-Dimensional Modeling from Reality” (Huber, 2002)
Heuristic Algorithm

Extract consistent spanning tree in model graph
Many spanning trees

Single incorrect match can destroy the maps
Inconsistent Loop Detection

Disambiguating Visual Relations Using Loop Constraints (Zach et al., CVPR 2010)

\[
\text{max} \sum_L \rho_L x_L \\
\text{s.t.} \quad x_L \geq x_e \quad \forall e \in L \\
\quad x_L \leq \sum_{e \in L} x_e \\
\quad x_L, x_e \in [0, 1]
\]

\(x_e = 1\) for false positive edge
\(x_L = \text{max of } x_e \text{ over loop}\)

Used to deal with repeating structures like windows!
An Optimization Approach to Improving Collections of Shape Maps

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Abstract

Finding an informative, structure-preserving map between two shapes has been a long-standing problem in computer graphics, involving a variety of solution approaches and applications. However, in many cases, given not only two related shapes, but a collection of them, and considering each pairwise map independently, we do not take full advantage of all existing information. For example, a notorious problem with computing shape maps is the ambiguity in the map, when there exist two maps, say $\mathbf{m}_{i,j}$ and $\mathbf{m}_{j,i}$, based on the sensitivity to how the map is chosen in the neighborhood. Given the context map consistency, we can choose solutions that help us replace a source interpolation problem with an optimization problem that个城市 individually for each shape, as long as the maps are for improving maps.

Categories and Subject Descriptors

1. Introduction

Definition 3: Given a collection of maps $\mathcal{M}$, let $\mathcal{B}(\mathcal{M}) = \{\mathbf{m}_{i,j} \in \mathcal{M} \mid E_{\text{acc}}(\mathbf{m}_{i,j}) > 0\}$ — the collection of inaccurate maps. Then we say that $\mathcal{M}$ is almost accurate, if there do not exist two maps $\mathbf{m}_1, \mathbf{m}_2 \in \mathcal{B}(\mathcal{M})$, which both belong to the same 3-cycle in $G_\mathcal{M}$. We call such maps isolated.

Iteratively fix triplets and reweight
Exploring Collections of 3D Models using Fuzzy Correspondences (Kim et al., SIGGRAPH 2012)
Fuzzy Correspondences: Idea

- Compute \( N_k \times N_k \) similarity matrix
  - Same number of samples per surface
  - Align similar shapes

- Compute spectral embedding

- Use as descriptor: Display \( e^{-|d_i-d_j|^2} \)
Global optimization to choose among many possible segmentations

“Joint Shape Segmentation with Linear Programming”
(Huang, Koltun, Guibas; SIGGRAPH Asia 2011)
Joint Segmentation: Motivation

Structural similarity of segmentations

- Extraneous geometric clues

Single shape segmentation [Chen et al. 09]

Joint shape segmentation [Huang et al. 11]

“Joint Shape Segmentation with Linear Programming”
(Huang, Koltun, Guibas; SIGGRAPH Asia 2011; slides provided by authors)
Joint Shape Segmentation: Motivation

Structural similarity of segmentations

- Low saliency

Single shape segmentation
[Chen et al. 09]

Joint shape segmentation
[Huang et al. 11]

“Joint Shape Segmentation with Linear Programming”
(Huang, Koltun, Guibas; SIGGRAPH Asia 2011; slides provided by authors)
Joint Segmentation: Motivation

(Rigid) invariance of segments

- Articulated structures

  Single shape segmentation
  [Chen et al. 09]

  Joint shape segmentation
  [Huang et al. 11]

“Joint Shape Segmentation with Linear Programming”
(Huang, Koltun, Guibas; SIGGRAPH Asia 2011; slides provided by authors)
Initial subsets of randomized segmentations

“Joint Shape Segmentation with Linear Programming”
(Huang, Koltun, Guibas; SIGGRAPH Asia 2011; slides provided by authors)
Each point covered by one segment

$$|\text{cover}(p)| = 1 \quad \forall p \in W$$

Avoid tiny segments

$$\text{score}(S') = \sum_{s \in S} \text{area}(s) \cdot \text{repetitions}_s$$

"Joint Shape Segmentation with Linear Programming"
(Huang, Koltun, Guibas; SIGGRAPH Asia 2011; slides provided by authors)
Consistency Term

- Defined in terms of mappings
  - Oriented
  - Partial

Many-to-one correspondences

Partial similarity

“Joint Shape Segmentation with Linear Programming”
(Huang, Koltun, Guibas; SIGGRAPH Asia 2011; slides provided by authors)
Multi-Way Joint Segmentation

- Objective function

\[ \sum_{i=1}^{n} \text{score}(S_i) + \sum_{(S_i, S_j) \in \mathcal{E}} \text{consistency}(S_i, S_j) \]

See paper: Linear program relaxation
Q: Can you extract consistent maps in a globally optimal way?
Basic Setup

Map as a permutation matrix

\[
\begin{pmatrix}
1 & 0 & 0 & \cdots & 0 \\
0 & 0 & 0 & \cdots & 1 \\
0 & 1 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & 0 \\
0 & 0 & 1 & \cdots & 0
\end{pmatrix}
\]
What is the inverse of a permutation matrix?
### Discrete Relaxation

A map can be represented as a doubly-stochastic matrix, where each row and column sums to 1.

$$
\begin{pmatrix}
1 & 0 & 0 & \cdots & 0 \\
0 & 0 & 0 & \cdots & 1 \\
0 & 1 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 1 & \cdots & 0
\end{pmatrix}
$$

**Sums to 1**

Map as a doubly-stochastic matrix
Basic Setting

- Given \( n \) objects
- Each object sampled with \( m \) points

“Consistent Shape Maps via Semidefinite Programming” (Huang & Guibas, SGP 2013)
Map Collection: Matrix Representation

\[ X = \begin{bmatrix}
I_m & X_{12} & \cdots & X_{1n} \\
X_{12}^T & I_m & \cdots & \vdots \\
\vdots & \vdots & \ddots & X_{(n-1),n} \\
X_{1n}^T & X_{(n-1),n}^T & \cdots & I_m
\end{bmatrix} \]

- Diagonal blocks are identity matrices
- Off diagonal blocks are permutation matrices
- Symmetric
What is the rank of a consistent map collection matrix?
Hint: “Urshape” Factorization

\[
X = \begin{bmatrix}
I_m & X_{12} & \cdots & X_{1n} \\
X_{12}^T & I_m & \cdots & \\
\vdots & \vdots & \ddots & X_{(n-1),n} \\
X_{1n}^T & X_{(n-1),n}^T & \cdots & I_m \\
\end{bmatrix}
\]

- Diagonal blocks are identity matrices
- Off diagonal blocks are permutation matrices
- Symmetric
Rank $m$, Number of Samples

\[ X_{ij} = X_{j1}^{\top} X_{i1} \iff X = \begin{pmatrix} I_m \\ \vdots \\ X_{n1}^{\top} \end{pmatrix} \begin{pmatrix} I_m & \cdots & X_{n1} \end{pmatrix} \]
Many Equivalent Conditions

Definition 2.1 Given a shape collection $\mathcal{S} = \{S_1, \ldots, S_n\}$ of $n$ shapes where each shape consists of the same number of samples, we say a map collection $\Phi = \{\phi_{ij} : S_i \rightarrow S_j | 1 \leq i, j \leq n\}$ of maps between all pairs of shapes is cycle consistent if and only if the following equalities are satisfied:

- $\phi_{ii} = id_{S_i}, \quad 1 \leq i \leq n \quad \text{(1-cycle)}$
- $\phi_{ij} \circ \phi_{ji} = id_{S_i}, \quad 1 \leq i < j \leq n \quad \text{(2-cycle)}$
- $\phi_{ki} \circ \phi_{jk} \circ \phi_{ij} = id_{S_i}, \quad 1 \leq i < j < k \leq n \quad \text{(3-cycle)}$

where $id_{S_i}$ denotes the identity self-map on $S_i$.

Equivalence for binary map matrix $\Phi$:

1. $\Phi$ is cycle-consistent

2. $X = Y_i^\top Y_i$, where $Y_i = (X_{i1}, \ldots, X_{in})$

3. $X \succeq 0$
Equivalence for binary map matrix $\Phi$: 

1. $\Phi$ is cycle-consistent 

2. $X = Y_i^\top Y_i$, where $Y_i = (X_{i1}, \ldots, X_{in})$ 

3. $X \succeq 0$
Approximation by Consistent Maps

\[
\begin{align*}
\max_X & \quad \sum_{i,j \in E} \langle X_{ij}^{in}, X_{ij} \rangle \\
\text{s.t.} & \quad X \in \{0, 1\}^{nm \times nm} \\
& \quad X \succeq 0 \\
& \quad X_{ii} = I_m \\
& \quad X_{ij} \mathbf{1} = 1 \\
& \quad X_{ij}^\top \mathbf{1} = 1
\end{align*}
\]
Approximation by Consistent Maps

\[
\begin{align*}
\max_{X} & \quad \sum_{i,j \in E} \langle X_{in}^{ij}, X_{ij} \rangle \\
\text{s.t.} & \quad X \in \{0, 1\}^{nm \times nm} \\
& \quad X \succeq 0 \\
& \quad X_{ii} = I_m \\
& \quad X_{ij} \mathbf{1} = 1 \\
& \quad X_{ij}^T \mathbf{1} = 1
\end{align*}
\]

Maximize number of preserved matches
Approximation by Consistent Maps

\[
\begin{align*}
\max_{X} & \quad \sum_{i,j \in E} \langle X_{ij}^{in}, X_{ij} \rangle \\
\text{s.t.} & \quad X \in \{0, 1\}^{nm \times nm} \\
& \quad X \succeq 0 \\
& \quad X_{ii} = I_m \\
& \quad X_{ij} 1 = 1 \\
& \quad X_{ij}^T 1 = 1
\end{align*}
\]
Approximation by Consistent Maps

\[ \max_X \sum_{i,j \in E} \langle X_{ij}^{in}, X_{ij} \rangle \]

s.t.

\[ X \in \{0, 1\}^{nm \times nm} \]

\[ X \geq 0 \]

\[ X_{ii} = I_m \]

\[ X_{ij} 1 = 1 \]

\[ X_{ij}^T 1 = 1 \]

Every block is a permutation
Approximation by Consistent Maps

$$\max_{X} \sum_{i,j \in E} \langle X_{ij}^{in}, X_{ij} \rangle$$

s.t. \( X \in \{0, 1\}^{nm \times nm} \)

\( X \succeq 0 \)

\( X_{ii} = I_m \) \( \text{(Self maps are identity)} \)

\( X_{ij} \mathbf{1} = \mathbf{1} \)

\( X_{ij}^\top \mathbf{1} = \mathbf{1} \)
Approximation by Consistent Maps

$$\max_X \sum_{ij \in E} \langle X_{ij}^{in}, X_{ij} \rangle$$

s.t. \hspace{1cm} X \in \{0, 1\}^{nm \times nm}

\hspace{1cm} X \succeq 0

\hspace{1cm} X_{ii} = I_m

\hspace{1cm} X_{ij} 1 = 1

\hspace{1cm} X_{ij}^\top 1 = 1

Already showed:
Equivalent to low-rank
Approximation by Consistent Maps

$$\max_X \sum_{i,j \in E} \langle X_{ij}^{in}, X_{ij} \rangle$$

s.t. $X \in \{0, 1\}^{nm \times nm}$

$X \succeq 0$

$X_{ii} = 1_m$

$X_{ij} 1 = 1$

$X_{ij}^T 1 = 1$
Convex Relaxation

\[
\begin{align*}
\max_X & \quad \sum_{i,j \in E} \langle X_{ij}^{in}, X_{ij} \rangle \\
\text{s.t.} & \quad X \geq 0 \\
& \quad X \geq 0 \\
& \quad X_{ii} = I_m \\
& \quad X_{ij} \mathbf{1} = \mathbf{1} \\
& \quad X_{ij}^T \mathbf{1} = \mathbf{1}
\end{align*}
\]
Rounding Procedure

\[ \text{max } x \quad \langle X, X_0 \rangle \]
\[ \text{s.t.} \quad X \geq 0 \]
\[ X \mathbf{1} = 1 \]
\[ X^\top \mathbf{1} = 1 \]

Linear assignment problem
Can tolerate $\lambda_2/4(n - 1)$ incorrect correspondences from each sample on one shape.

$\lambda_2$ is algebraic connectivity; bounded above by two times maximum degree
Can tolerate 25% incorrect correspondences from each sample on one shape.

$\lambda_2$ is algebraic connectivity; bounded above by two times maximum degree.
Phase Transition

Always recovers / Never recovers
Example Result
Solving the multi-way matching problem by permutation synchronization

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Abstract

The problem of matching not just two, but m different sets of objects to each other arises in many contexts, including finding the correspondence between feature points across multiple images in computer vision. At present it is usually solved by matching the sets pairwise, in series. In contrast, we propose a new method, Permutation Synchronization, which finds all the matchings jointly, in one shot, via a relaxation to eigenvector decomposition. The resulting algorithm is both computationally efficient, and, as we demonstrate with theoretical arguments as well as experimental results, much more stable to noise than previous methods.

1 Introduction

Finding the correct bijection between \( x_1', x_2', \ldots, x_m' \) is a fundamental problem in many contexts [1]. In this paper, we consider its relation to the permutation \( X_1, X_2, \ldots, X_m \). Our primary motivation is in the landmark problem (feature points) across a sequence of images. Consider the problem of matching multiple images. Each image has \( m \) feature points \( x_1', x_2', \ldots, x_m' \). We are interested in finding a permutation \( X \) that maximizes the number of matched feature points.

Presently, multi-matching is usually solved by matching each image to others one-by-one. In general, this approach is computationally expensive and does not take advantage of the structure of the data. Instead, we propose a new method, Permutation Synchronization, which finds all the matchings jointly, in one shot, via a relaxation to eigenvector decomposition. The resulting algorithm is both computationally efficient and, as we demonstrate with theoretical arguments and experimental results, much more stable to noise than previous methods.
Q: Where do the pairwise input maps come from?
Possible Extension with Guarantees

Tight Relaxation of Quadratic Matching

Koby Bazerman\thanks{Equal contribution.}, Shabbar Z. Kovalsky\thanks{Equal contribution.}, Ronen Basri, Yaron Lipman

Weizmann Institute of Science

Abstract

Establishing point correspondences between shapes is extremely challenging as it involves both finding sets of semantically persistent feature points, as well as their combinatorial matching. We focus on the latter and consider the Quadratic Assignment Matching (QAM) model. We suggest a novel convex relaxation for this NP-hard problem that builds upon a rank-one reformulation of the problem in a higher dimension, followed by relaxation into a semidefinite program (SDP). Our method is shown to be a certain hybrid of the popular spectral and doubly-stochastic relaxations of QAM and in particular we prove that it is tighter than both.

Figure 1: Consistent Collection Matching. Results of the proposed one-stage procedure for finding consistent correspondences between shapes in a collection showing strong variability and non-rigid deformations.

\[
\begin{align}
\max_Y & \; \text{tr}(WY) \\
\text{s.t.} & \; Y \geq [X][X]^T \\
& \; X \in \text{conv } \Pi_n^k \\
& \; \text{tr} Y = k \\
& \; Y \geq 0 \\
& \sum_{q,s,t} Y_{qrst} = k^2 \quad \text{if } q=s, r \neq t \\
& \quad = 0, \quad \text{if } r = t, q \neq s \\
& \quad = \min \{X_{qrst}, X_{srqt}\}, \quad \text{otherwise} \tag{7g}
\end{align}
\]

\[
\begin{align}
\max_{X,Y} & \; \sum_{i,j} \text{tr}(W^{ij}Y^{ij}) \\
\text{s.t.} & \; \left(X^{ij}, Y^{ij}\right) \in C^k \quad \forall i < j \\
& \; X^{ii} \in D \cap \text{conv } \Pi_n^k \quad \forall i \\
& \; X \geq 0 \tag{10d}
\end{align}
\]
Consistent Partial Matching of Shape Collections via Sparse Modeling

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Abstract

Recent efforts in the area of joint object matching approach the problem by taking as input the shapes which are then jointly optimized across the whole collection so that certain accuracy measure is satisfied. One natural requirement is cycle-consistency — namely the fact that maps of the same result regardless of the path taken in the shape collection. In this paper, we introduce a joint measure of metric distortion directly over the space of cycle-consistent maps. From this formalism, we formulate the problem as a series of quadratic programs, making our technique a natural candidate for analyzing collections with large or complex structures. The particular form of the problem allows us to leverage results and tools from the computer vision literature. This enables a highly efficient optimization procedure which allows access to solutions in a matter of minutes in collections with hundreds of shapes.

Categories and Subject Descriptors (according to ACM CCS): 1.3.5 [Computer Graphics]: Computational Geometry and Object Modeling—Shape Analysis

1. Introduction

Finding matches among multiple objects is a research topic that has been investigated for a long time. This is a natural and widely accepted criterion in cycle-consistency [2, 4, 9, 10], namely the composition of maps with respect to a pose. In this context, we propose a new framework that allows to efficiently find matches among large collections of objects. The proposed method is based on a sequence of quadratic programs; based on metric distortion and WKS descriptor match.
Approximate Methods

Multiplicative updates for nonconvex nonnegative matrix factorization

Entropic Metric Alignment for Correspondence Problems

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Abstract

Many shape and image processing tools rely on computation of correspondences between geometric domains. Efficient methods that stably extract “soft” matches in the presence of diverse geometric structures have proven to be valuable for shape retrieval and transfer of geometric information. With these applications in mind, we attempt to obtain a probabilistic correspondence that optimizes an entropic approximation of a global warping function. The method can be expressed as a simple binary program where the entropy term measures the degree to which a solution is consistent with a global warping. We demonstrate experiments illustrating the convergence of our algorithm to a variety of graphs. We validate the performance of our algorithm to a variety of graphs.

Figure 1: Entropic GW can find correspondences between a source surface (left) and a surface with similar structure, a surface with shared semantic structure, a noisy 3D point cloud, an icon, and a hand drawing. Each fuzzy map was computed using the same code.

In this paper, we propose a new correspondence algorithm that minimizes distortion of long and short-range distances alike. We study an entropically-regularized version of the Gromov-Wasserstein (GW) mapping objective function from [Mémoli 2011] measuring the distortion of geodesic distances. The optimizer is in a probabilistic matching expressed as a “fuzzy” correspondence matrix in the style of [Kim et al. 2012; Solomon et al. 2012].

In this paper, we propose a new correspondence algorithm that minimizes distortion of long and short-range distances alike. We study an entropically-regularized version of the Gromov-Wasserstein (GW) mapping objective function from [Mémoli 2011] measuring the distortion of geodesic distances. The optimizer is in a probabilistic matching expressed as a “fuzzy” correspondence matrix in the style of [Kim et al. 2012; Solomon et al. 2012].

sinh

The primary factor that distinguishes correspondence algorithms is the choice of objective functions. Different choices of objective functions express different notions of which correspondences are “desirable.” Classical theorems from differential geometry and modern algorithms consider local distortion, producing maps that take tangent planes to tangent planes with as little stretch as possible, while allowing “farmer’s field” maps that are more efficient.
Computer Vision Perspective

Unpaired Image-to-Image Translation using Cycle-Consistent Adversarial Networks

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Berkeley AI Research (BAIR) laboratory, UC Berkeley

Figure 1: Given any two unordered image collections $X$ and $Y$, our algorithm learns to automatically “translate” an image from one into the other and vice versa: (left) Monet paintings and landscape photos from Flickr; (center) zebras and horses from ImageNet; (right) summer and winter Yosemite photos from Flickr. Example application (bottom): using a collection of paintings of famous artists, our method learns to render natural photographs into the respective styles.

Abstract
Image-to-image translation is a class of vision and graphics problems where the goal is to learn the mapping

1. Introduction
What did Claude Monet see as he placed his paintbrush to his canvas? How did Van Gogh capture the moment? How did Cézanne observe his subject? How did Ukiyo-e artists in Japan render their subjects?

Slides courtesy the authors
https://junyanz.github.io/CycleGAN/
Paired vs. Unpaired Problems

**Paired**

\[ x_i , y_i \]

\[
\begin{align*}
\{ & \text{Cat, } y_i \\
& \text{Cat, } y_i \\
& \text{Cat, } y_i \\
\} \\
\{ & \text{Cat, } y_i \\
\} \\
\} \\
\vdots \\
\end{align*}
\]

**Unpaired**

\[ X , Y \]

\[
\begin{align*}
\{ & \text{Horse} \\
& \text{Horse} \\
& \text{Horse} \\
\} \\
\{ & \text{Zebra} \\
\} \\
\} \\
\vdots \\
\end{align*}
\]
Adversarial Networks: Problem

- **Input (X)**
  - Connected to **Generator (G)**
  - **Generated Image (G(x))**
  - **Discriminator (D)**
    - 
      - **Real Image**
      - **Real Too!**

- **Input (X)**
  - Connected to **Generator (G)**
  - **Generated Image (G(x))**
  - **Discriminator (D)**
    - 
      - **Real Image**
      - **Real Too!**
Mode collapse
Cycle-Consistent Adversarial Networks

\[
\|F(G(x)) - x\|_1
\]

Reconstruction error

Large cycle loss

Small cycle loss

[Zhu*, Park*, Isola, and Efros, ICCV 2017]
Cycle Consistency Loss

\[ x \xrightarrow{G} y \xrightarrow{F} x \xrightarrow{G} y \]

\[ D_Y(G(x)) \]

\[ \| F(G(x)) - x \|_1 \]

\[ D_G(F(x)) \]

\[ \| G(F(y)) - y \|_1 \]

See similar formulations [Yi et al. 2017], [Kim et al. 2017]

[Zhu*, Park*, Isola, and Efros, ICCV 2017]
More Than Two Domains?

(a) Cross-domain models

(b) StarGAN

Choi et al., CVPR 2018
Extra: Angular Synchronization

Justin Solomon

6.838: Shape Analysis
Spring 2021