Vector Fields: Introduction

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6.838: Shape Analysis
Spring 2021
Vector Field Processing on triangle meshes

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Pixar Animation Studios

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Michigan State University

Render the Possibilities
SIGGRAPH 2016

Check out course notes!
Directional Field Synthesis, Design, and Processing

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Abstract

Directional fields and vector fields play an increasingly important role in computer graphics and geometry processing. The synthesis of directional fields on surfaces, or other spatial domains, is a fundamental step in numerous applications, such as mesh generation, deformation, texture mapping, and many more. The wide range of applications resulted in definitions for many types of directional fields: from vector and tensor fields, over line and cross fields, to frame and vector-set fields. Depending on the application at hand, researchers have used various notions of objectives and constraints to synthesize such fields. These notions are defined in terms of fairness, feature alignment, symmetry, or field topology, to mention just a few. To facilitate these objectives, various representations, discretizations, and optimization strategies have been developed. These choices come with varying strengths and weaknesses. This report provides a systematic overview of directional field synthesis for graphics applications, the challenges it poses, and the methods developed in recent years to address these challenges.

Categories and Subject Descriptors (according to ACM CCS): 1.3.5 [Computer Graphics]: Computational Geometry and Object Modeling—

1. Introduction

There have been significant developments in directional field synthesis over the past decade. These developments have been driven...
Why Vector Fields?
Why Vector Fields?

Simulation and PDE
Why Vector Fields?

"Blood flow in the rabbit aortic arch and descending thoracic aorta"
Vincent et al.; J. Royal Society 2011

Biological science and imaging
Why Vector Fields?

https://disc.gsfc.nasa.gov/featured-items/airs-monitors-cold-weather

Weather modeling
Vectorization of Line Drawings via Polyvector Fields (Bessmeltsev & Solomon; TOG 2019)
Why Vector Fields?

Simulation and engineering
Why Vector Fields?

"OT-Flow: Fast and Accurate Continuous Normalizing Flows via Optimal Transport" (Onken et al.)

Continuous normalizing flows
Why Vector Fields?

Pastry design
Many Challenges

- Directional derivative?
- Purely intrinsic version?
- Singularities?
- Flow lines?
- ...

- How to discretize?
- Discrete derivatives?
- Singularity detection?
- Flow line computation?
- ...

Theoretical

Discrete
Plan

Crash course in theory/discretization of vector fields.
Many Challenges

- Directional derivative?
- Purely intrinsic version?
- Singularities?
- Flow lines?
- How to discretize?
- Discrete derivatives?
- Singularity detection?
- Flow line computation?

Theoretical

Discrete
Recall:

Tangent Space

\( T_p M = \gamma'(0), \) where \( \gamma(0) = p \)

\( = \text{image}(dg_p) \)

\( M \subseteq \mathbb{R}^n \)

\( g \)

\( U \subseteq \mathbb{R}^m \)
Some Definitions

Tangent bundle:
\[ TM := \{(p, v) : v \in T_p M\} \]

Vector field:
\[ u : M \to TM \text{ with } u(p) = (p, v), v \in T_p M \]
Scalar Functions

Map points to real numbers
**Definition** (Differential). Suppose \( \varphi : M \to N \) is a map from a submanifold \( M \subseteq \mathbb{R}^k \) into a submanifold \( N \subseteq \mathbb{R}^\ell \). Then, the differential \( d\varphi_p : T_p M \to T_{\varphi(p)} N \) of \( \varphi \) at a point \( p \in M \) is given by

\[
d\varphi_p(v) := (\varphi \circ \gamma)'(0),
\]

where \( \gamma : (-\epsilon, \epsilon) \to M \) is any curve with \( \gamma(0) = p \) and \( \gamma'(0) = v \in T_p M \).
**Proposition**  For each \( p \in M \), there exists a unique vector \( \nabla f(p) \in T_p M \) so that \( df_p(v) = v \cdot \nabla f(p) \) for all \( v \in T_p M \).
Q: How do you differentiate a vector field?
Common point of confusion.
(especially for your instructor)
THE THING IS .... ITS COMPlicated
What’s the issue?

How to identify different tangent spaces?
Many Notions of Derivative

- **Differential of covector**
  (defer for now) (or, forever?)

- **Lie derivative**
  Weak structure, purely topological

- **Covariant derivative**
  Strong structure, involves geometry
$D_X Y \triangleq \lim_{t \to 0} \frac{Y(p + tX) - Y(p)}{t}$
Vector Field Flows: Diffeomorphism

\[ \frac{d}{dt} \psi_t = V \circ \psi_t \]

Useful property: \( \psi_{t+s}(x) = \psi_t(\psi_s(x)) \)

Diffeomorphism with inverse \( \psi_{-t} \)
Fun example:

Killing Vector Fields (KVFs)

Preserves distances infinitesimally

Wilhelm Killing
1847-1923
Germany
Differential of Vector Field Flow

\[ d\psi_t(p) : T_p M \to T_{\psi_t(p)} M \]
$$\left( \mathcal{L}_V W \right)_p := \lim_{t \to 0} \frac{1}{t} \left[ (d\psi_{-t})_{\psi_t(p)}(W_{\psi_t(p)}) - W_p \right]$$
It’s pronounced "Lee"
Not "Lahy" or "Lye"

(BTW: It’s “oiler,” not “you-ler”)

\[(\mathcal{L}_V W)_p := \lim_{t \to 0} \frac{1}{t} \left[ (d\psi_{-t})_{\psi_t(p)}(W_{\psi_t(p)}) - W_p \right] \]
What We Want

“What is the derivative of the orange vector field in the blue direction?”

What we don’t want:
Specify blue direction anywhere but at $p$. 
Parallel Transport

Canonical identification of tangent spaces
\[ \nabla_v w := [dw(v)]^\parallel = \text{proj}_{T_p\mathcal{M}}(w \circ \alpha)'(0) \]

Synonym: (Levi-Civita) Connection

Integral curve of $v$ through $p$

\[ [dw(v)]^\perp = \Pi(v, w)n \]
Some Properties

Properties of the Covariant Derivative

As defined, $\nabla_Y Y$ depends only on $V_p$ and $Y$ to first order along $c$.

Also, we have the Five Properties:

1. $C^\infty$-linearity in the $V$-slot:
   \[ \nabla_{V_1 + aV_2} Y = \nabla_{V_1} Y + f \nabla_{V_2} Y \text{ where } f : S \to \mathbb{R} \]

2. $\mathbb{R}$-linearity in the $Y$-slot:
   \[ \nabla_V (Y_1 + aY_2) = \nabla_V Y_1 + a \nabla_V Y_2 \text{ where } a \in \mathbb{R} \]

3. Product rule in the $Y$-slot:
   \[ \nabla_V (f Y) = f \cdot \nabla_V Y + (\nabla_V f) \cdot Y \text{ where } f : S \to \mathbb{R} \]

4. The metric compatibility property:
   \[ \nabla_V \langle Y, Z \rangle = \langle \nabla_V Y, Z \rangle + \langle Y, \nabla_V Z \rangle \]

5. The "torsion-free" property:
   \[ \nabla_{V_1} V_2 - \nabla_{V_2} V_1 = [V_1, V_2] \]

The Lie bracket

\[ [V_1, V_2](f) := D_{V_1}D_{V_2}(f) - D_{V_2}D_{V_1}(f) \]

Defines a vector field, which is tangent to $S$ if $V_1, V_2$ are!
Challenge Problem

4-3. In your study of differentiable manifolds, you have already seen another way of taking “directional derivatives of vector fields,” the Lie derivative $\mathcal{L}_X Y$.

(a) Show that the map $\mathcal{L} : \mathcal{T}(M) \times \mathcal{T}(M) \to \mathcal{T}(M)$ is not a connection.

(b) Show that there is a vector field on $\mathbb{R}^2$ that vanishes along the $x^1$-axis, but whose Lie derivative with respect to $\partial_1$ does not vanish on the $x^1$-axis. [This shows that Lie differentiation does not give a well-defined way to take directional derivatives of vector fields along curves.]
Recall:

Geodesic Equation

\[ \text{proj}_{T\gamma(s)M} [\gamma''(s)] \equiv 0 \]

- The only acceleration is out of the surface
  - No steering wheel!
Intrinsic Geodesic Equation

\[ \nabla \ddot{\gamma}(t) \ddot{\gamma}(t) = 0 \]

- No stepping on the accelerator
  - No steering wheel!
Parallel Transport

0 = \nabla_{\dot{\gamma}(t)} V

Only path-independent if domain is flat.

Preserves length, inner product
(can be used to define covariant derivative)
Holonomy

Integrated Gaussian curvature

Path dependence of parallel transport
2D Vector Field Topology

Drawings by Jonas Kibelbek
Poincaré-Hopf Theorem

\[ \sum_{i} \text{index}_{x_i} (v) = \chi(M) \]

where vector field \( v \) has isolated singularities \( \{x_i\} \).

\[ v(c(t)) = \|v(c(t))\| \begin{pmatrix} \cos \alpha(t) \\ \sin \alpha(t) \end{pmatrix} \]

Image from “Directional Field Synthesis, Design, and Processing” (Vaxman et al., EG STAR 2016)
Famous Corollary

Hairy ball theorem
## Extension in 2D: Direction Fields

<table>
<thead>
<tr>
<th>Field Type</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>1-vector field</td>
<td>One vector, classical “vector field”</td>
</tr>
<tr>
<td>2-direction field</td>
<td>Two directions with $\pi$ symmetry, “line field”, “2-RoSy field”</td>
</tr>
<tr>
<td>1³-vector field</td>
<td>Three independent vectors, “3-polyvector field”</td>
</tr>
<tr>
<td>4-vector field</td>
<td>Four vectors with $\pi/2$ symmetry, “non-unit cross field”</td>
</tr>
<tr>
<td>4-direction field</td>
<td>Four directions with $\pi/2$ symmetry, “unit cross field”, “4-RoSy field”</td>
</tr>
<tr>
<td>2²-vector field</td>
<td>Two pairs of vectors with $\pi$ symmetry each, “frame field”</td>
</tr>
<tr>
<td>2²-direction field</td>
<td>Two pairs of directions with $\pi$ symmetry each, “non-ortho. cross field”</td>
</tr>
<tr>
<td>6-direction field</td>
<td>Six directions with $\pi/3$ symmetry, “6-RoSy”</td>
</tr>
<tr>
<td>2³-vector field</td>
<td>Three pairs of vectors with $\pi$ symmetry each</td>
</tr>
</tbody>
</table>

“Directional Field Synthesis, Design, and Processing” (Vaxman et al., EG STAR 2016)
Designing N-PolyVector Fields with Complex Polynomials

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Abstract

We introduce N-PolyVector fields, a generalization of N-RoSy fields for which the vectors are neither necessarily orthogonal nor orientation symmetric. We formally define a novel representation for N-PolyVectors as the root sets of complex polynomials and analyze their topological and geometric properties. A smooth N-PolyVector field can be efficiently generated by solving a sparse linear system without integer variables. We exploit the flexibility of N-PolyVector fields to design conjugate vector fields, offering an intuitive tool to generate planar quadrilateral meshes.

One encoding of direction fields

\begin{align*}
\{u_0, u_1, \ldots, u_k\} \\
\longmapsto f(z) &= (z - u_0) \cdots (z - u_k) \\
\longmapsto f(z) &= z^{k+1} + a_k z^k + \cdots + a_1 z + a_0 \\
\longmapsto \{a_0, \ldots, a_k\}
\end{align*}
What singular structures are possible?

What is the relationship between meshes and fields?

Image from:
Hex Mesh Singular Structures

Images from:
Field-Guided Meshing Pipeline


Frame per element on a tet mesh
Example Frame Fields

Images from:
Nine spherical harmonic coefficients per point

Original idea in [Huang et al. 2011]
Visualization from [Ray, Sokolov, and Lévy 2016]

\[ f(x, y, z) = x^4 + y^4 + z^4 \]
\[ f(x, y, z) = x^4 + y^4 + z^4 \]

\{ \text{rotations of } f(x, y, z) \} \n\not\subset \n\{ \text{degree-4 polynomials} \}
More Careful Characterization

Representation Theory Perspective

Space of rotations $\text{SO}(3)$ \xrightarrow{\rho} \text{GL}(9)$

Wigner d-matrices $\cdot q_0$

Octahedral variety $\mathcal{F}$ \xrightarrow{\phi} F$

Isometry (up to scale)

Orbit of $f$

Roughly: Coefficients of $f(R^\top x)$
Extension: Odeco Frames

$$\sum_{\tilde{i}} \lambda_{\tilde{i}} (v_{\tilde{i}}^\top x)^d$$

Orthogonally-decomposable tensors
Vanishing near singular curves
Why Odeco?

Octahedral

Energy density

Odeco
Many Challenges

- Directional derivative?
- Purely intrinsic version?
- Singularities?
- Flow lines?
- ...

- How to discretize?
- Discrete derivatives?
- Singularity detection?
- Flow line computation?
- ...

Theoretical

Discrete
No consensus:

- Triangle-based
- Edge-based
- Vertex-based
Vector Fields on Triangle Meshes

*No consensus:*
- Triangle-based
- Edge-based
- Vertex-based

\[ \subseteq \mathbb{R}^3 \]
Triangle-Based

- Triangle as its **own tangent plane**
- One vector per triangle
  - Piecewise constant
  - Discontinuous at edges/vertices
- Easy to unfold/hinge
Discrete Levi-Civita Connection

- Simple notion of parallel transport
- Transport around vertex:
  Excess angle is (integrated)
  **Gaussian curvature** (holonomy!)
Represent using angle $\theta_{edge}$ of extra rotation.
Trivial Connections

- Vector field design
- **Zero holonomy** on discrete cycles
  - Except for a few singularities
- Path-independent away from singularities

“Trivial Connections on Discrete Surfaces.” Crane et al., SGP 2010.
Trivial Connections: Details

- Solve $\theta_{edge}$ of extra rotation per edge

- Linear constraint:
  - Zero holonomy on basis cycles
    - $V+2g$ constraints: Vertex cycles plus harmonic
    - Fix curvature at chosen singularities

- Underconstrained: Minimize $||\vec{\theta}||$
  - “Best approximation” of Levi-Civita
Result

Resulting trivial connection
(no other singularities present)

Linear system
Extension:

Helmholtz-Hodge Decomposition

\[ \mathcal{R} \nabla f \]

Divergence free

\[ \nabla f \]

Curl free

2g-dimensional

Harmonic

Image courtesy K. Crane
Recall:

Gradient of a Hat Function

\[
\| \nabla f \| = \frac{1}{\ell_3 \sin \theta_3} = \frac{1}{h}
\]

\[
\nabla f = \frac{e_{23}}{2A}
\]

Length of \(e_{23}\) cancels "base" in \(A\)
Recall:

Euler Characteristic

\[ V - E + F := \chi \]

\[ \chi = 2 - 2g \]

\[ g = 0 \quad g = 1 \quad g = 2 \]
Discrete Helmholtz-Hodge

\[ 2 - 2g = V - E + F \]

\[ \implies 2F = (V - 1) + (E - 1) + 2g \]

Either
- Vertex-based gradients
- Edge-based rotated gradients

or
- Edge-based gradients
- Vertex-based rotated gradients

“Mixed” finite elements

Dimensionality works out perfectly!
Face-Based Calculus

**Vertex-based**

"Conforming"
Already did this in 6.838

**Edge-based**

"Nonconforming"
[Wardetzky 2006]

Relationship: \( \psi_{ij} = \phi_i + \phi_j - \phi_k \)

Gradient Vector Field
Volumetric Extension?

3D Hodge Decompositions of Edge- and Face-based Vector Fields

RUNDONG ZHAO, Michigan State University
MATHIEU DESBRUN, California Institute of Technology
GUO-WEI WEI and YIYING TONG, Michigan State University

Fig. 1. Five-Component Vector Field Decomposition. On a tetrahedral mesh of the kitten with a spherical cavity, a vector field is decomposed into a gradient field with zero potential on the boundary, a curl field with its vector potential orthogonal to the boundary, a pair of tangential and normal harmonic fields, and a harmonic field that is both a gradient and a curl field. Potential fields are shown in the corners of their corresponding components.

We present a compendium of Hodge decompositions of vector fields on tetrahedral meshes embedded in the 3D Euclidean space. After describing the foundations of the Hodge decomposition in the continuous setting, we describe how to implement a five-component orthogonal decomposition that generically splits, for a variety of boundary conditions, any given discrete vector field expressed as discrete differential forms into two potential fields, as well as three additional harmonic components that arise from the topology or boundary of the domain. The resulting decomposition is proper and mimetic, in the sense that the theoretical dualities on the kernel spaces of vector Laplacians valid in the continuous case (including correspondences to cohomology and homology groups) are exactly preserved in the discrete realm. Such a decomposition only involves simple linear algebra with symmetric matrices, and can thus serve as a basic computational tool for vector static and dynamical problems — for instance, fluid simulation to enforce incompressibility. The mathematical foundations behind such decompositions were developed using the theory of differential forms for any finite-dimensional compact manifold without boundary early on [Hodge 1941], but were fully extended to manifolds with boundaries much more recently [Shonkwiler 2009].

In computer graphics, the analysis and processing of vector fields over surfaces have received plenty of attention in recent years. Consequently, the resulting computational tools needed to achieve a Hodge decomposition have been well documented and tested on various applications; see, e.g., recent surveys on surface vector field analysis [Vaxman et al. 2016; de Goes et al. 2016a]. For the case...
Vector Fields on Triangle Meshes

No consensus:

- Triangle-based
- Edge-based
- Vertex-based

$\subseteq \mathbb{R}^3$

Defer to DEC!
Vector Fields on Triangle Meshes

No consensus:
- Triangle-based
- Edge-based
- Vertex-based
**Pros**
- Possibility of higher-order differentiation

**Cons**
- Vertices don’t have natural tangent spaces
- Gaussian curvature concentrated
2D (Planar) Case: Easy

Piecewise-linear \((x,y)\) components
\[ \|d\|_{KVF}^2 := \frac{1}{2} \int_{\Omega(x)} \|J_d + J_d^\top\|_{\text{Fro}}^2 dA = d^\top K(x) d \]

Example: Killing Energy

“AKVF:”
Approximate Killing Vector Field

“As-Killing-As-Possible Vector Fields for Planar Deformation” (SGP 2011) Solomon, Ben-Chen, Butscher, Guibas
Interpolate vector and tangent space!
Parallel transport radially from vertex

$r_i := \frac{2\pi}{2\pi - \kappa_i}$

Preserve radial lines (change their spacing)

"Vector Field Design on Surfaces," Zhang et al., TOG 2006

Provides notion of tangency, but continuity issues
Recent Method

Discrete Connection and Covariant Derivative for Vector Field Analysis and Design

Beibei Liu and Yiying Tong
Michigan State University
and
Fernando de Goes and Mathieu Desbrun
California Institute of Technology

In this paper, we introduce a discrete definition of connection on simplicial manifolds, involving closed-form continuous expressions within simplices and finite rotations across simplices. The finite-dimensional parameters of this connection are optimally computed by minimizing a quadratic measure of the deviation to the (discontinuous) Levi-Civita connection induced by the embedding of the input triangle mesh, or to any metric connection with arbitrary cone singularities at vertices. From this discrete connection, a covariant derivative is constructed through exact differentiation, leading to explicit expressions for local integrals of first-order derivatives (such as divergence, curl and the Cauchy-Riemann operator), and for $L_2$-based energies (such as the Dirichlet energy). We finally demonstrate the utility, flexibility, and accuracy of our discrete formulations for the design and analysis of vector, $n$-vector, and $n$-direction fields.

Categories and Subject Descriptors: 1.3.5 [Computer Graphics]: Computational Geometry & Object Modeling—Curve & surface representations.

CCS Concepts: Computing methodologies → Mesh models;

digital geometry processing, with applications ranging from texture synthesis to shape analysis, meshing, and simulation. However, existing discrete counterparts of such a differential operator acting on simplicial manifolds can either approximate local derivatives (such as divergence and curl) or estimate global integrals (such as the Dirichlet energy), but not both simultaneously.

In this paper, we present a unified discretization of the covariant derivative that offers closed-form expressions for both local and global first-order derivatives of vertex-based tangent vector fields on triangulations. Our approach is based on a new construction of discrete connections that provides consistent interpolation of tangent vectors within and across mesh simplices, while minimizing the deviation to the Levi-Civita connection induced by the 3D embedding of the input mesh—or more generally, to any metric connection with arbitrary cone singularities at vertices. We demonstrate the relevance of our contributions by providing new computational tools to design and edit vector and $n$-direction fields.
No consensus:
- Triangle-based
- Edge-based
- Vertex-based
No consensus:
- Triangle-based
- Edge-based
- Vertex-based
- ... others?
An Operator Approach to Tangent Vector Field Processing

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\(^3\)LIX, École Polytechnique

Figure 1: Using our framework various vector field design goals can be easily posed as linear constraints. Here, given three symmetry maps: rotational (\(S1\)), bilateral (\(S2\)) and front/back (\(S3\)), we can generate a symmetric vector field using only \(S1\) (left), \(S1 + S2\) (center) and \(S1 + S2 + S3\) (right). The top row shows the front of the 3D model, and the bottom row its back.

Abstract

In this paper, we introduce a novel coordinate-free method for manipulating and analyzing vector fields on discrete surfaces. Unlike the commonly used representations of a vector field as an assignment of vectors to the mesh, or as real values on edges, we argue that vector fields can also be naturally viewed as operators whose domain and range are functions defined on the mesh. Although this point of view is common in differential geometry.
Subdivision Exterior Calculus for Geometry Processing

Fernando de Goes, Pixar Animation Studios
Mathieu Desbrun, Caltech
Mark Meyer, Pixar Animation Studios
Tony DeRose, Pixar Animation Studios

Figure 1: Subdivision Exterior Calculus (SEC). We introduce a new technique to perform geometry processing applications on subdivision surfaces by extending Discrete Exterior Calculus (DEC) from the polygonal to the subdivision setting. With the presence of a few operators on the control mesh, SEC outperforms DEC in terms of accuracy in geometry with only minor computational overhead. For instance, while the spectral conformal parameterization [Mullen et al. 2008] of the control mesh of the mannequin head (left) results in large quasi-conformal distortion (mean = 1.784, max = 9.4) after subdivision (middle), simply substituting our SEC operators for the original DEC operators significantly reduces distortion (mean = 1.005, max = 3.0) (right). Parameters: shown at level 1 for clarity, exhibit substantial differences.

Abstract
This paper introduces a new computational method to solve differential equations on subdivision surfaces. Our approach adopts the numerical framework of Discrete Exterior Calculus (DEC) from the polygonal to the subdivision setting by exploiting the refinability of subdivision basis functions. The resulting Subdivision Exterior Calculus (SEC) provides significant improvements in accuracy compared to existing polygonal techniques, while offering exact finite-dimensional analogs of continuum structural identities such as Stokes’ theorem and Helmholtz-Hodge decomposition. We demonstrate the versatility and efficiency of SEC on common geometric processing tasks including parameterization, geodesic distance computations, and vector field design.

Keywords: Subdivision surfaces, discrete exterior calculus, discrete differential geometry, geometry processing.

Concepts: Mathematics of computing → Discretization; Computations in infinite fields.

Subdivision Directional Fields

BRAM CUSTERS, Utrecht University/TU Eindhoven
AMIR VAXMAN, Utrecht University

Fig. 1. Rotationally-seamless parameterization with a subdivision directional field. An initial field (left) is optimized for low curl at the coarsest level f = 0. We subdivide the field to fine level f = 3 (center), and then solve for a seamless parameterization in both levels (right). Our subdivision preserves curl, and then results in a low integration error in both levels. The coarse-level optimization takes 7.9 secs, the subdivision 7.6 secs, and the parameterization 7.6 secs, to a total of 22.1 secs. This is a speedup of about two orders of magnitude compared to running the curl optimization directly on the fine level, taking 1438.7 secs.

We present a novel linear subdivision scheme for face-based tangent directional fields on triangle meshes. Our subdivision scheme is based on a novel coordinate-free representation of directional fields as halfedge-based scalar quantities, bridging the mixed finite-element representation with discrete exterior calculus. By computing with differential operators, our subdivision is structure-preserving: it reproduces curl-free fields precisely, and preserves divergence-free fields in the weak sense. Moreover, our subdivision scheme directly extends to directional fields with several vectors per face by working on the branched covering space. Finally, we demonstrate how our scheme can be applied to directional-field design, advection, and robust earth mover’s distance computation, for efficient and robust computation.

CCS Concepts: → Computing methodologies → Mesh models; Mesh geometry models; Shape analysis;

Additional Key Words and Phrases: Directional Fields, Vector Fields, Subdivision Surfaces, Geometry Processing; Finite Element Methods; Discrete Exterior Calculus.

1 INTRODUCTION

Directional fields are central objects in geometry processing. They represent flows, alignments, and symmetry on discrete meshes. They are used for diverse applications such as meshing, fluid simulation, texture synthesis, architectural design, and many more.

There is then great value in devising robust and reliable algorithms that design and analyze such fields. In this paper, we work with piecewise-constant tangent directional fields, defined on the faces of a triangle mesh. A directional field is the assignment of several vectors per face, where the most commonly-used fields comprise single vectors. The piecewise-constant face-based representation of directional fields is a mainstream representation within the (mixed) finite-element method (FEM), where the vectors are often gradients of piecewise-linear functions spanned by values on the vertices.

Working with a fine-resolution mesh (red and yellow mesh)
Extra: Continuous Normalizing Flows

Justin Solomon

6.838: Shape Analysis
Spring 2021