Reversible Harmonic Maps

Danielle Ezuz (Technion)
Justin Solomon (MIT)
Mirela Ben-Chen (Technion)

ACM Transactions on Graphics 2019
Our Approach

Input: a sparse set of landmarks \((p_i, q_i)\)
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- Initialize the map by mapping **geodesic cells** of each landmark \(p_i\) to the corresponding landmark \(q_i\):
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Input: a sparse set of landmarks \((p_i, q_i)\)

• Initialize the map by mapping geodesic cells of each landmark \(p_i\) to the corresponding landmark \(q_i\)

• Optimize the map with respect to an energy that promotes smoothness and bijectivity
Discrete Dirichlet Energy

Measures **smoothness** of a map:

\[ E(\phi_{12}) = \frac{1}{2} \int_{M_1} |d\phi_{12}|^2 \]

A map is **harmonic** if it is a critical point of the Dirichlet energy
Discrete Dirichlet Energy

\[ E_D(\phi_{12}) = \sum_{(u,v) \in E_1} w_{uv} d_{M_2}^2(\phi_{12}(u), \phi_{12}(v)) \]
Discrete Precise Maps

Stochastic matrices with barycentric coordinates at each row:

\[ P_{12} = \begin{pmatrix} j & k & l \\ \vdots & -0.1 & 0.2 & -0.7 & \vdots \\ \end{pmatrix} \text{ row } i \]
Discrete Precise Maps

Stochastic matrices with barycentric coordinates at each row:

\[
\begin{pmatrix}
  j & k & l \\
  \vdots & \vdots & \vdots \\
  0.1 & 0.2 & 0.7 \\
\end{pmatrix}
\begin{pmatrix}
  v_j \\
  v_k \\
  v_l \\
\end{pmatrix}
\]

\[P_{12} \cdot V_2\]

\[V_2 \in \mathbb{R}^{n_2 \times 3}\] is a matrix with vertex coordinates of \(M_2\)
Discretization – Dirichlet Energy

If we replace the geodesic distances by Euclidean distances, the discrete Dirichlet energy is:

\[ E_D^{Euc}(P_{12}) = \| P_{12}V_2 \|_{W_1}^2 = Trace((P_{12}V_2)^T W_1 P_{12}V_2) \]

\( W_1 \) is a matrix with \(-w_{ij}\) at entry \(i, j\), and the sum of the weights on the diagonal
Discrete Dirichlet Energy

We use a *high dimensional embedding* where Euclidean distances approximate geodesic distances (MDS)

\[ X_2 \in \mathbb{R}^{n_2 \times 8} \]

Then the discrete Dirichlet energy is approximated by:

\[ E_D(P_{12}) = \| P_{12}X_2 \|_{W_1}^2 \]
Minimizing the Dirichlet Energy

A map that maps all vertices to a single point is harmonic.

Minimizing the harmonic energy “shrinks” the map:

Initial map (Id) 

optimize $E_D$
Reversibility

• We add a reversibility term to prevent the map from shrinking
Reversibility

Continuous setting:

\[ E_R(T_{12}, T_{21}) = \sum_{v \in V_1} d_{M_2}(v, T_{21}(T_{12}(v))) + \sum_{v \in V_2} d_{M_1}(v, T_{12}(T_{21}(v))) \]

The term \( E_R(T_{12}, T_{21}) \) promotes injectivity and surjectivity
Reversibility

Discrete setting:

\[ E_R(P_{12}, P_{21}) = \| P_{21}P_{12}X_2 - X_2 \|_{M_2}^2 + \| P_{12}P_{21}X_1 - X_1 \|_{M_1}^2 \]

Again we use \( X_1, X_2 \) the high dimensional embedding of each shape to approximate geodesic distances
Total Energy

We combine the Dirichlet energy and the reversibility term:

\[ E(P_{12}, P_{21}) = \alpha E_D(P_{12}) + \alpha E_D(P_{21}) + (1 - \alpha) E_R(P_{12}, P_{21}) \]

The parameter \( \alpha \) controls the trade off between the terms
Optimization

All the terms are quadratic, but $P_{12}, P_{21}$ are constrained to the feasible set of precise maps

$$P_{12} = \begin{pmatrix} j & k & l \\
\vdots & \ddots & \vdots \\
-0.1 & -0.2 & -0.7 \end{pmatrix}_{\text{row } i}$$

$$E(P_{12}, P_{21}) = \alpha E_D(P_{12}) + \alpha E_D(P_{21}) + (1 - \alpha) E_R(P_{12}, P_{21})$$
Optimization

We know how to optimize functions of the form:

$$\arg \min_{P_{12} \in S} \|P_{12}A - B\|^2$$

$S$ is the feasible set of precise maps
Optimization

If we constrain to \textbf{vertex-to-vertex} maps (subset of feasible set):

\( P_{12} \) is a binary stochastic matrix

\[
P_{12}^* = \arg \min_{P_{12} \in \mathcal{S}} \| P_{12} A - B \|_{M_1}^2
\]
Optimization

If $P_{12}$ is any **precise** map:

\[
\begin{pmatrix}
1 \\
1 \\
1 \\
1
\end{pmatrix}
\begin{pmatrix}
P_{12} & A
\end{pmatrix}
\]

\[
\min_{f \in F_2} \min_{b \geq 0, \Sigma b = 1} \langle b^\top, f \rangle
\]

\[
P^*_1 = \arg \min_{P_{12} \in S} \|P_{12}A - B\|^{2}_{M_1}
\]
Optimization

If $P_{12}$ is any **precise** map:

$$
\min \min_{f \in F_2} \min_{b \geq 0, \Sigma b = 1} \left\| (b^\top) \left( \begin{array}{c}
\text{Rows of } f \\
\end{array} \right) - (i) \right\|_{M_1}
$$

Seems expensive

- Optimize barycentric coordinates by projecting the $i_{th}$ row to a triangle in $\mathbb{R}^{k_2}$ (geometric algorithm)

- Parallelizable!

$$
P_{12}^* = \arg \min_{P_{12} \in S} \|P_{12} A - B\|_{M_1}^2
$$
Optimization

Our energies are not of this form exactly:

\[ E_D(P_{12}) = \text{Tr}((P_{12}X_2)^TW_1P_{12}X_2) \]

\[ E_R(P_{12}, P_{21}) = \|P_{21}P_{12}X_2 - X_2\|_{M_2}^2 + \|P_{12}P_{21}X_1 - X_1\|_{M_1}^2 \]

Should not depend on \( P_{12} \)

We use “half quadratic splitting” such that our energy is of the desired form
Optimization

Introduce new variables

- $X_{12}$ should approximate $P_{12}X_2$, so we add a term $\|P_{12}X_2 - X_{12}\|^2$
- $X_{21}$ should approximate $P_{21}X_1$, so we add a term $\|P_{21}X_1 - X_{21}\|^2$

We replace $P_{12}X_2$ by $X_{12}$ wherever it bothers our optimization
Optimization

We rewrite our energies with the new variables:

\[ E_D(X_{12}) = Tr(X_{12}^T W_1 X_{12}) \]

\[ E_R(X_{12}, X_{21}, P_{12}, P_{21}) = \| P_{21} X_{12} - X_2 \|_{M_2}^2 + \| P_{12} X_{21} - X_1 \|_{M_1}^2 \]

\[ E_Q(X_{12}, P_{12}) = \| P_{12} X_2 - X_{12} \|_{M_1}^2 \]
Optimization

We optimize the energy:

\[ E(X_{12}, X_{21}, P_{12}, P_{21}) = \alpha E_D(X_{12}) + \alpha E_D(X_{21}) + \]
\[ + (1 - \alpha) E_R(X_{12}, X_{21}, P_{12}, P_{21}) + \]
\[ + \beta E_Q(X_{12}, P_{12}) + \beta E_Q(X_{21}, P_{21}) \]

by alternatingly optimizing for each variable

• Optimize \( P_{12} \) or \( P_{21} \) using projection
• Optimize \( X_{12} \) or \( X_{21} \) by solving a linear system
Results – SHREC’07

Target  Hyperbolic Orbifolds  Weighted Averages  Ours
Results – SHREC’07

Target
Hyperbolic Orbifolds
Weighted Averages
Ours
Results

Interpolation methods require the source and target shape to have a matching triangulation, that can be computed by our method

Heeren, Behrend, Martin Rumpf, Peter Schröder, Max Wardetzky and Benedikt Wirth, “Exploring the Geometry of the Space of Shells”, 2014