Reversible Harmonic Maps

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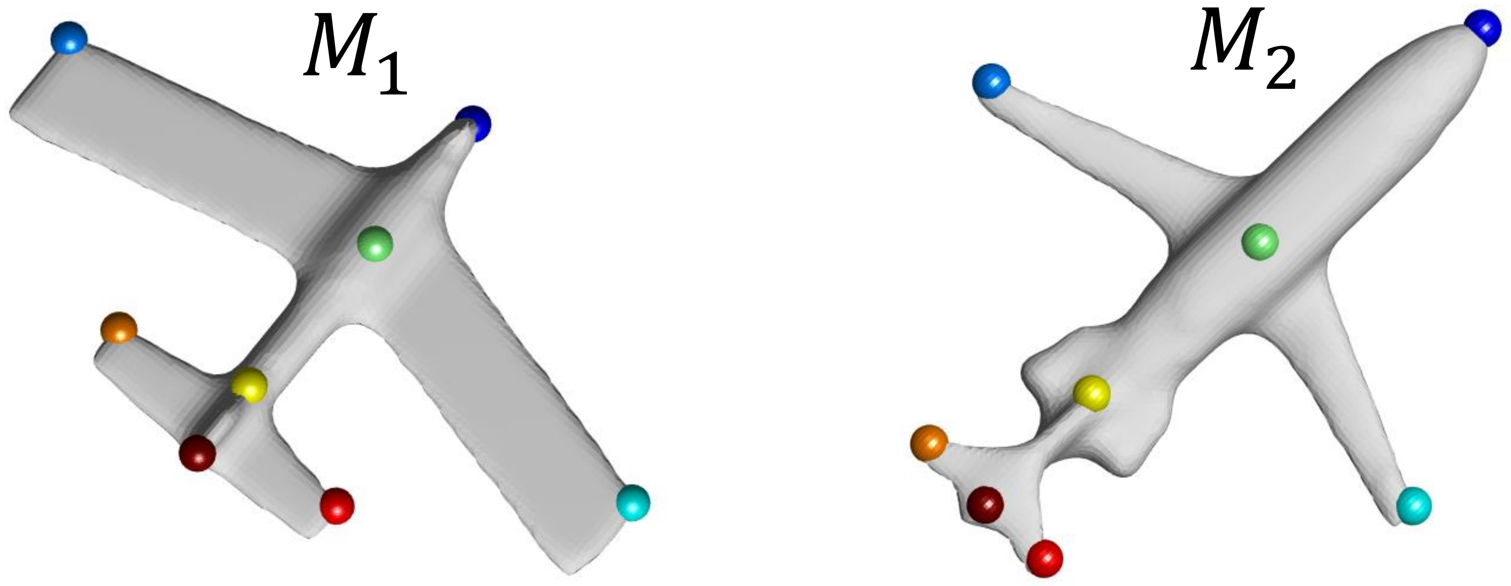
Our Approach

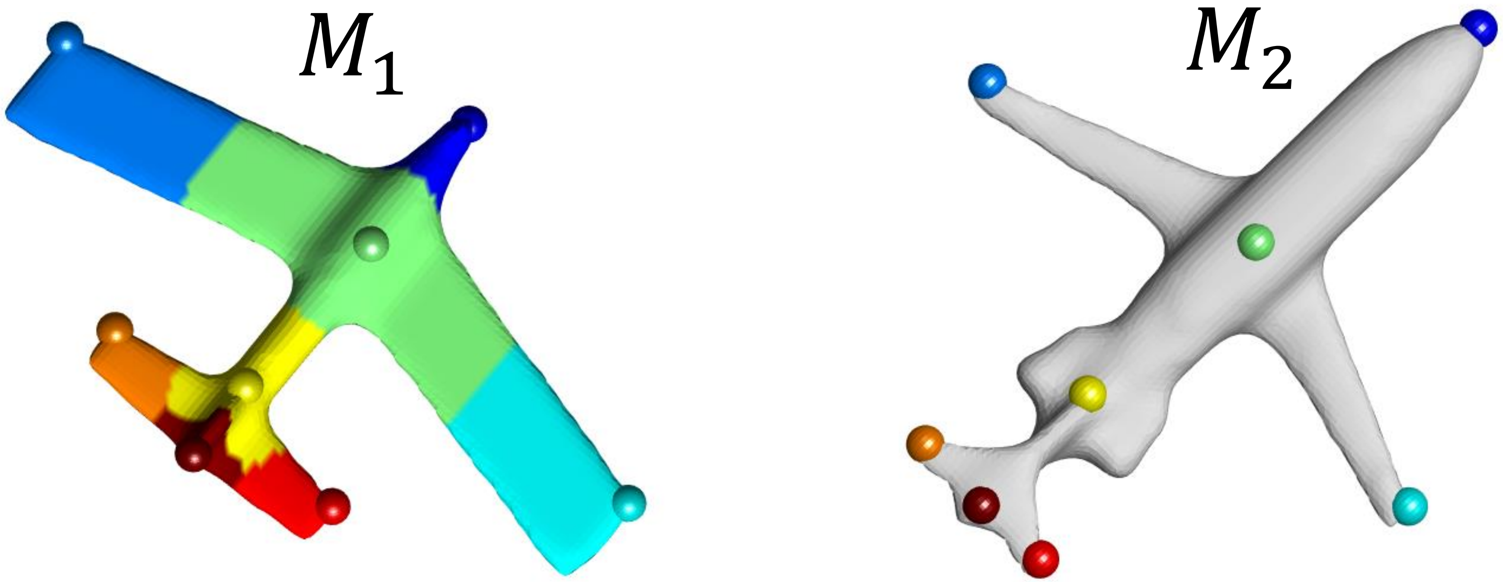
Input: a sparse set of landmarks (p_i, q_i)



Our Approach

Input: a sparse set of landmarks (p_i, q_i)

- Initialize the map by mapping **geodesic cells** of each landmark p_i to the corresponding landmark q_i:



Our Approach

Input: a sparse set of landmarks (p_i, q_i)

- Initialize the map by mapping **geodesic cells** of each landmark p_i to the corresponding landmark q_i

- Optimize the map with respect to an **energy** that promotes **smoothness** and **bijectivity**

Discrete Dirichlet Energy

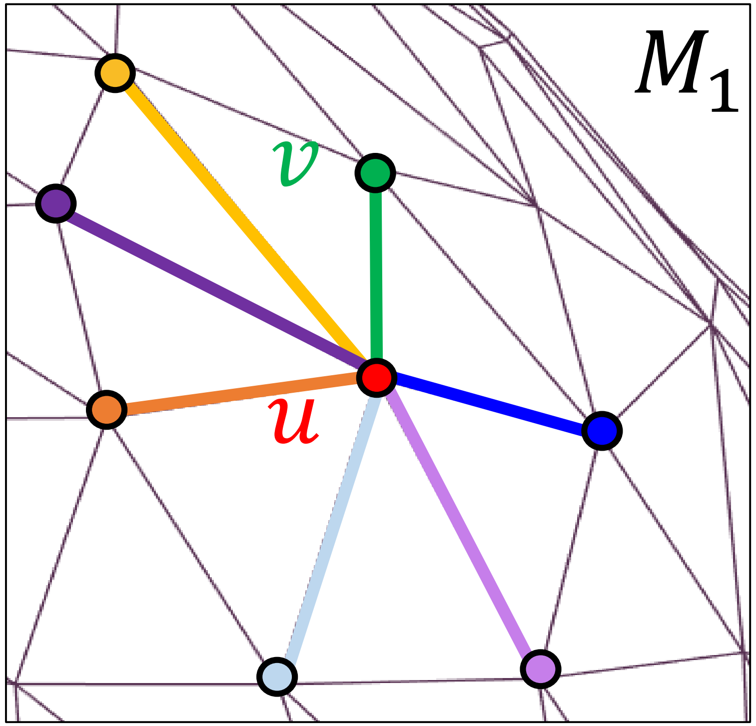
Measures **smoothness** of a map:

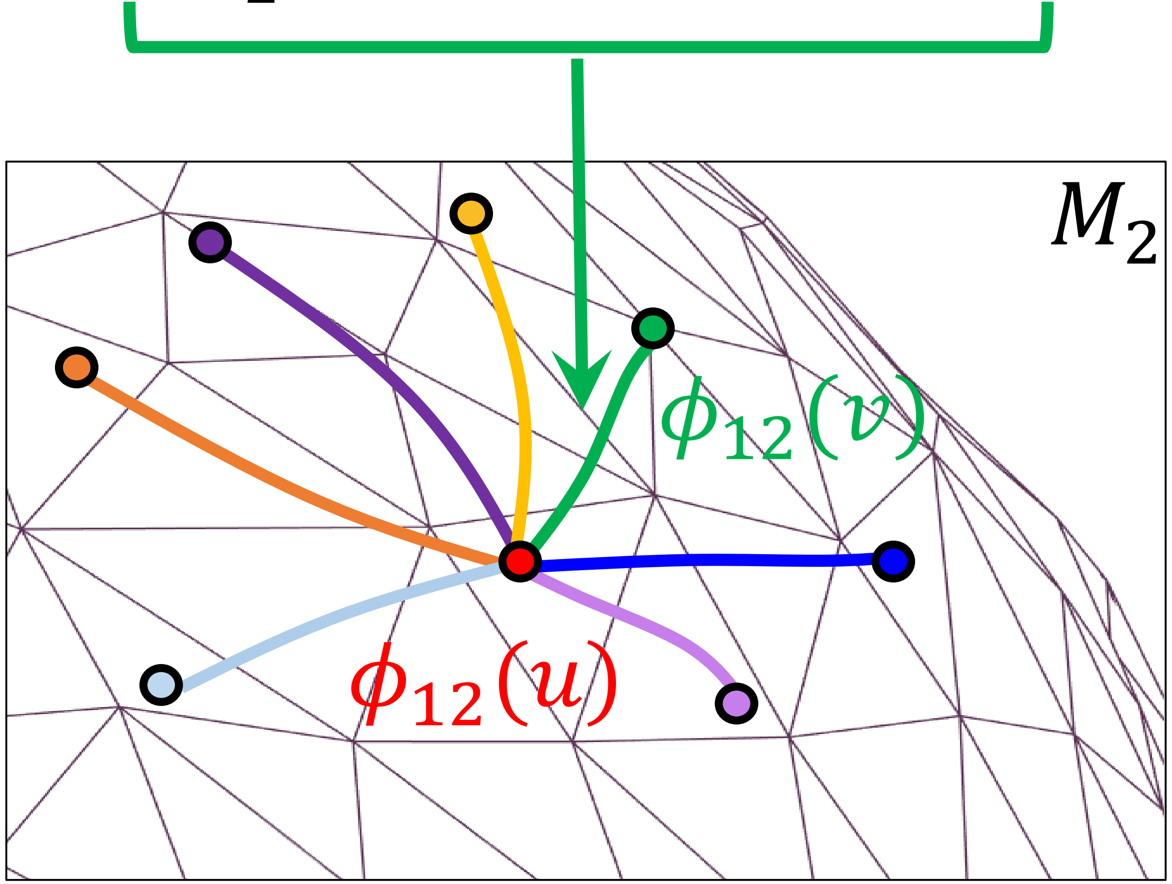
$$E(\phi_{12}) = \frac{1}{2} \int_{M_1} |d\phi_{12}|^2$$

A map is **harmonic** if it is a critical point of the Dirichlet energy

Discrete Dirichlet Energy

$$E_D(\phi_{12}) = \sum_{(u,v)\in E_1} w_{uv} d^2_{M_2}(\phi_{12}(u), \phi_{12}(v))$$

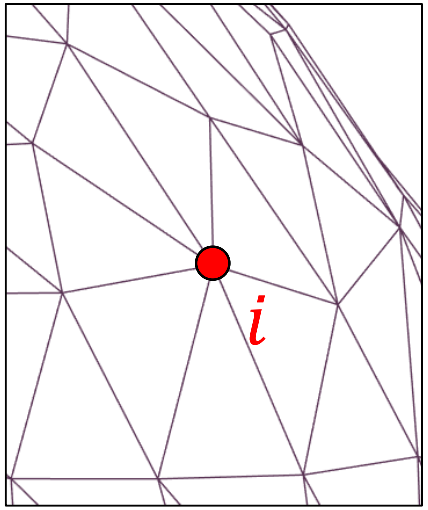




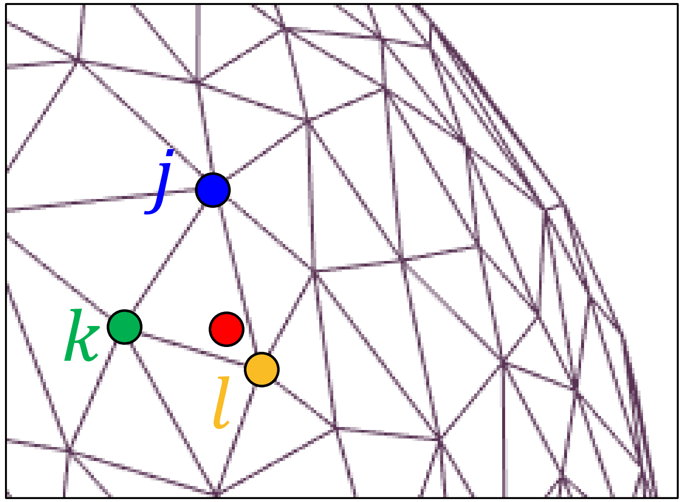
Discrete Precise Maps

Stochastic matrices with barycentric coordinates at each row:

$$P_{12} = \begin{pmatrix} & j & & k & & l & \\ & & & \vdots & & & \\ - & 0.1 & - & 0.2 & - & 0.7 & - \\ & & & \vdots & & & \end{pmatrix} \text{row } i$$



M_1

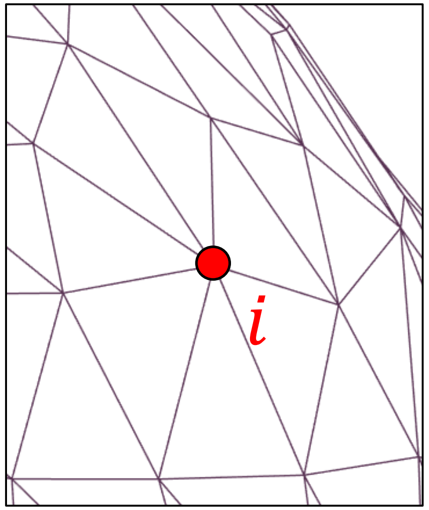


M_2

Discrete Precise Maps

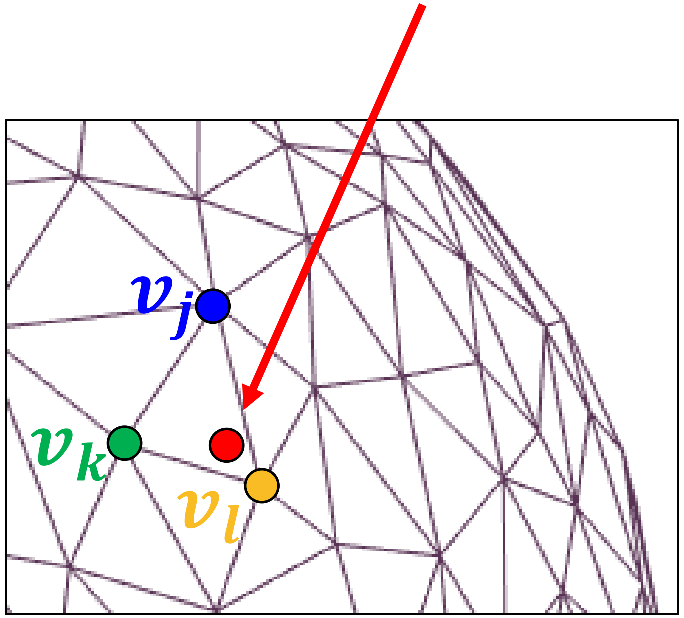
Stochastic matrices with barycentric coordinates at each row:

$$\underbrace{\begin{pmatrix} & j & k & l & \\ & & \vdots & & \\ i & 0.1 & 0.2 & 0.7 & \\ & & \vdots & & \end{pmatrix}}_{P_{12}} \underbrace{\begin{pmatrix} - v_j - \\ - v_k - \\ - v_l - \end{pmatrix}}_{V_2}$$



M_1

i^{th} row of $P_{12}V_2$



M_2

$V_2 \in \mathbb{R}^{n_2 \times 3}$ is a matrix with vertex coordinates of M_2

Discretization – Dirichlet Energy

If we replace the geodesic distances by Euclidean distances, the discrete Dirichlet energy is:

$$E_D^{Euc}(P_{12}) = \|P_{12}V_2\|_{W_1}^2 = Trace\left((P_{12}V_2)^\top W_1 P_{12} V_2\right)$$

W_1 is a matrix with $-w_{ij}$ at entry i, j, and the sum of the weights on the diagonal

$$\begin{array}{c c} & \begin{array}{ccc} j & i & k \end{array} \\ i & \left(\begin{array}{ccc} -w_{ij} & \sum_v w_{iv} & -w_{ik} \\ & & \\ & & \end{array}\right) \end{array}$$

$$W_1$$

Discrete Dirichlet Energy

We use a *high dimensional embedding* where Euclidean distances approximate geodesic distances (MDS)

$$X_2 \in \mathbb{R}^{n_2 \times 8}$$

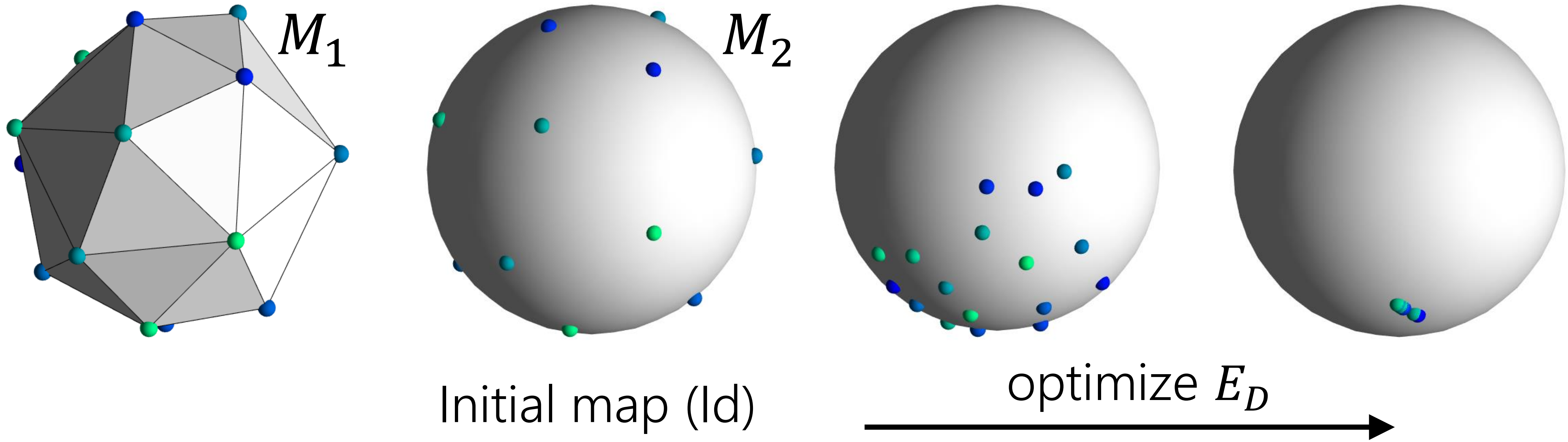
Then the discrete Dirichlet energy is approximated by:

$$E_D(P_{12}) = ||P_{12}X_2||_{W_1}^2$$

Minimizing the Dirichlet Energy

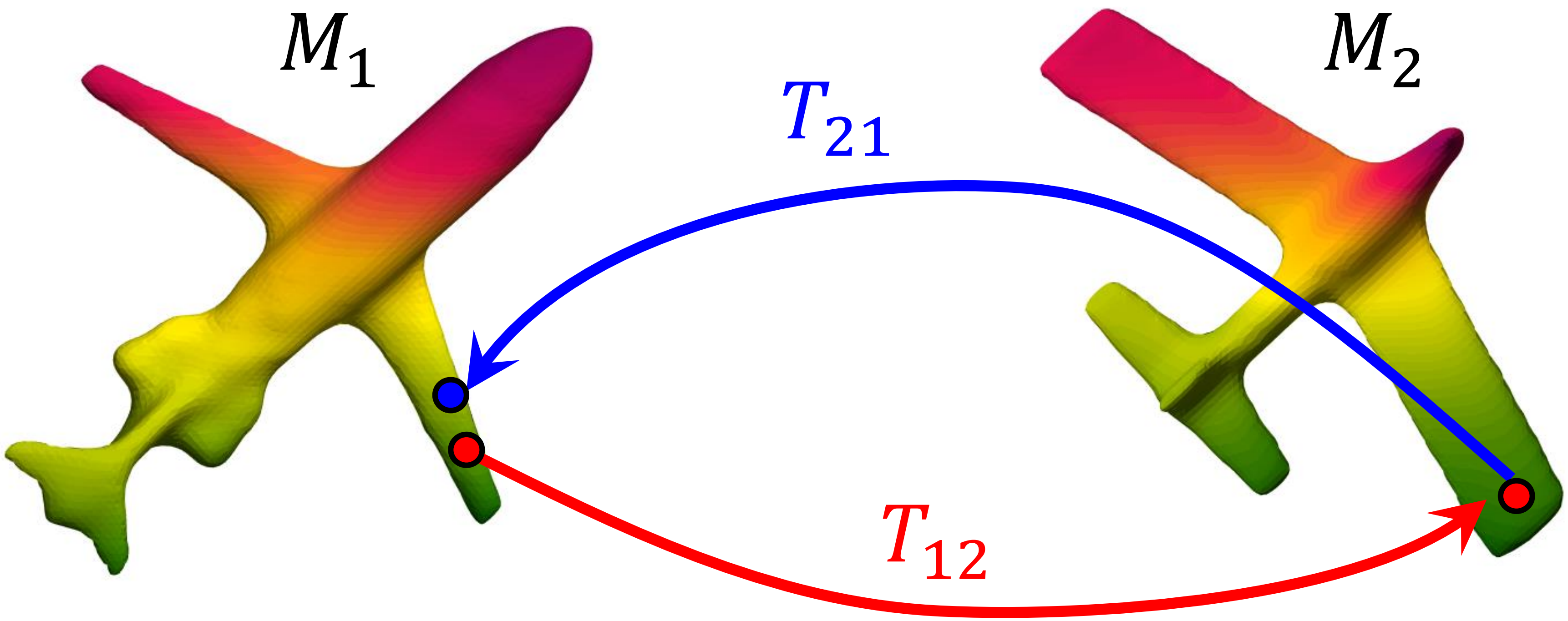
A map that maps all vertices to a single point is harmonic

Minimizing the harmonic energy “shrinks” the map:

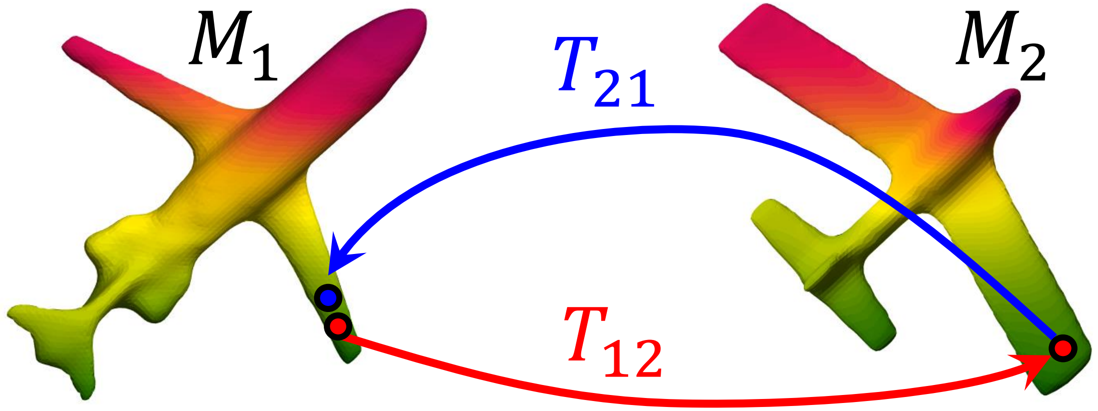


Reversibility

- We add a **reversibility term** to prevent the map from shrinking



Reversibility

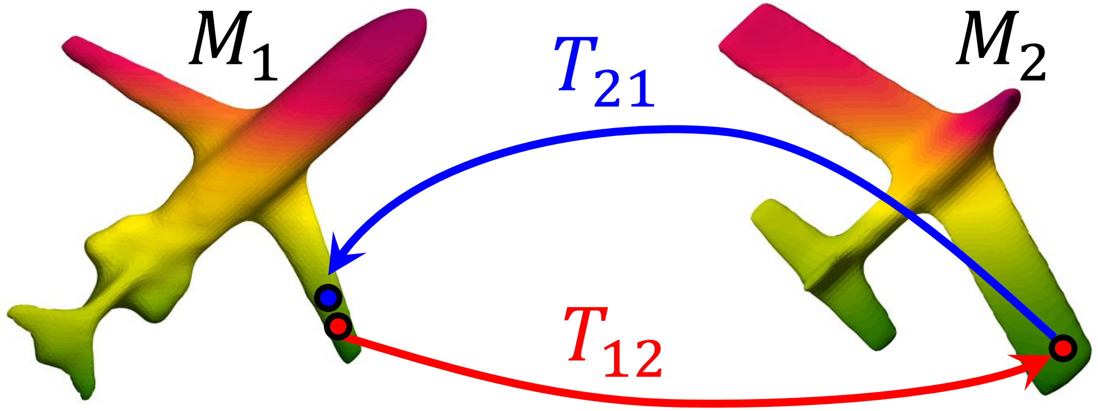


Continuous setting:

$$E_R(T_{12}, T_{21}) = \sum_{v \in V_1} d_{M_2}\left(v, T_{21}(T_{12}(v))\right) + \sum_{v \in V_2} d_{M_1}\left(v, T_{12}(T_{21}(v))\right)$$

The term $E_R(T_{12}, T_{21})$ promotes *injectivity* and *surjectivity*

Reversibility



Discrete setting:

$$E_R(P_{12}, P_{21}) = \|P_{21}P_{12}X_2 - X_2\|_{M_2}^2 + \|P_{12}P_{21}X_1 - X_1\|_{M_1}^2$$

Again we use X_1, X_2 the high dimensional embedding of each shape to approximate geodesic distances

Total Energy

We combine the Dirichlet energy and the reversibility term:

$$E(P_{12}, P_{21}) = \alpha E_D(P_{12}) + \alpha E_D(P_{21}) + (1 - \alpha) E_R(P_{12}, P_{21})$$

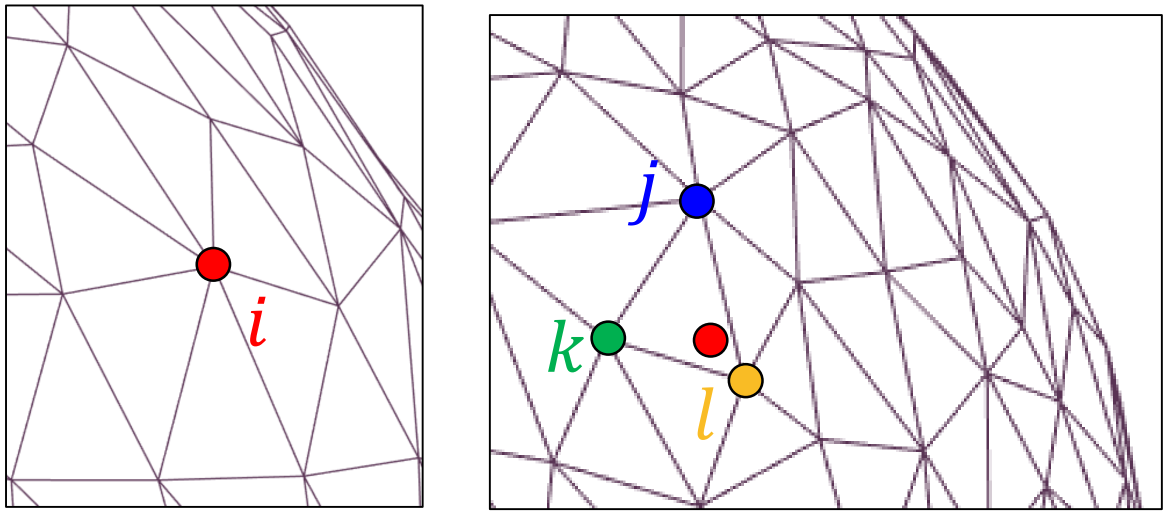
The parameter α controls the trade off between the terms

$$E(P_{12}, P_{21}) = \alpha E_D(P_{12}) + \alpha E_D(P_{21}) + (1-\alpha) E_R(P_{12}, P_{21})$$

Optimization

All the terms are quadratic, but P_{12}, P_{21} are constrained to the feasible set of precise maps

$$P_{12} = \begin{pmatrix} & j & & k & & l & \\ & & & \vdots & & & \\ - & 0.1 & — & 0.2 & — & 0.7 & — \\ & & & \vdots & & & \end{pmatrix} \text{row } i$$



M_1 M_2

Optimization

We know how to optimize functions of the form:

$$\arg\min_{\mathrm{P}_{12} \in S} \|P_{12}A - B\|^2$$

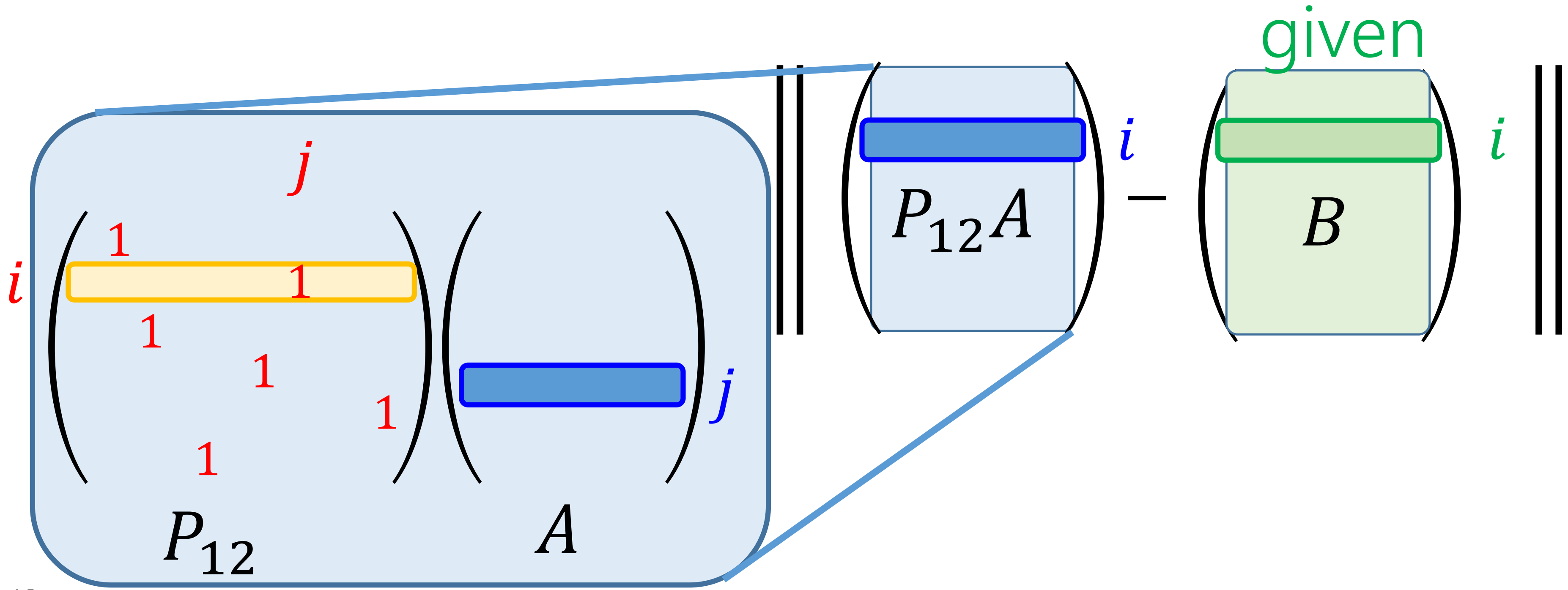
S is the feasible set of precise maps

Optimization

$$P_{12}^* = \arg\min_{P_{12} \in S} \|P_{12}A - B\|_{M_1}^2$$

If we constrain to **vertex-to-vertex** maps (subset of feasible set):

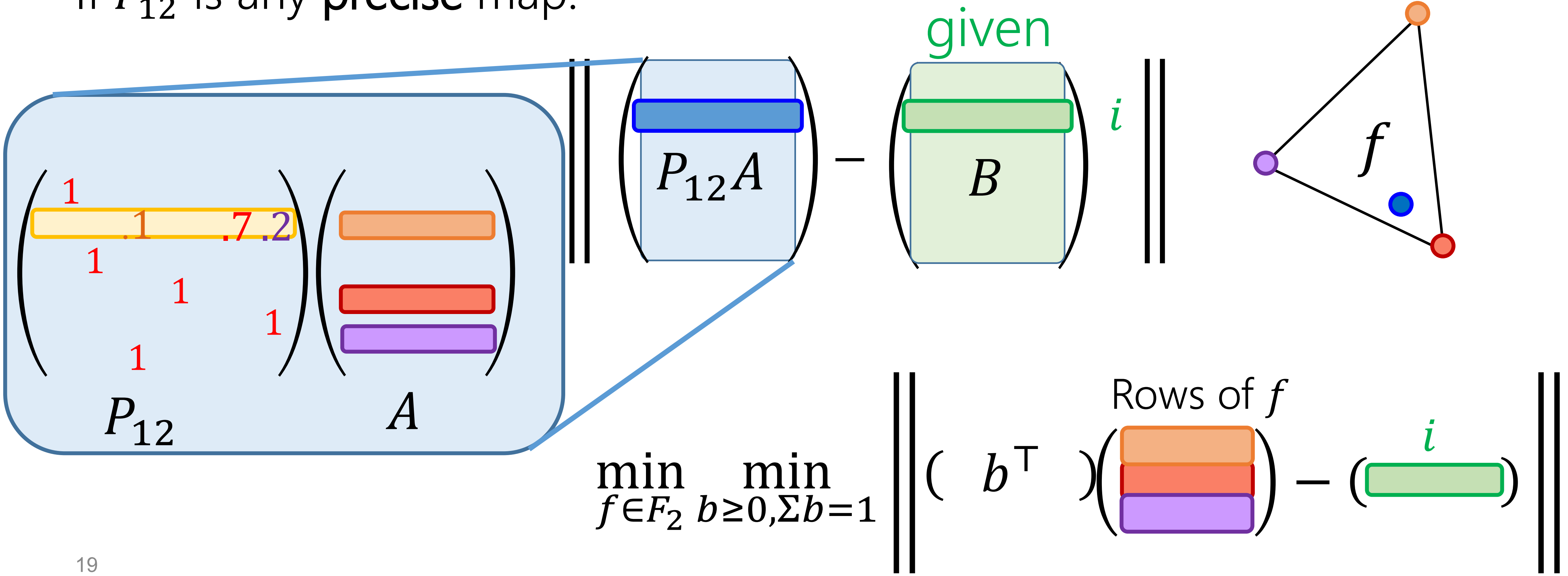
P_{12} is a binary stochastic matrix



Optimization

$$P_{12}^* = \arg\min_{P_{12}\in S} \|P_{12}A - B\|_{M_1}^2$$

If P_{12} is any **precise** map:

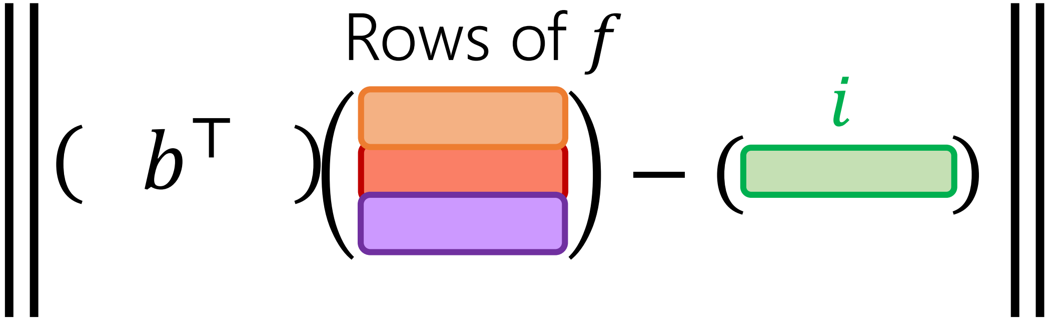


Optimization

$$P_{12}^{*} = \arg\min_{P_{12}\in S} \|P_{12}A - B\|_{M_1}^2$$

If P_{12} is any **precise** map:

$$\min_{f\in F_2} \min_{b\geq 0, \Sigma b=1}$$



Seems expensive

- Optimize barycentric coordinates by projecting the i_{th} row to a triangle in $\mathbb{R}^{k_2}$ (geometric algorithm)
- Parallelizable!

Optimization

Our energies are not of this form exactly:

$$E_D(P_{12}) = Tr\big((P_{12}X_2)^\top W_1 P_{12} X_2\big)$$

$$E_R(\mathrm{P}_{12}, P_{21}) = \|P_{21}\underbrace{P_{12}X_2}_{\text{Should not depend on } P_{12}} - X_2\|_{M_2}^2 + \|P_{12}P_{21}X_1 - X_1\|_{M_1}^2$$

We use "half quadratic splitting" such that our energy is of the desired form

Optimization

Introduce new variables

- X_{12} should approximate $P_{12}X_2$, so we add a term $\|P_{12}X_2 - X_{12}\|^2$
- X_{21} should approximate $P_{21}X_1$, so we add a term $\|P_{21}X_1 - X_{21}\|^2$

We replace $P_{12}X_2$ by X_{12} wherever it bothers our optimization

Optimization

We rewrite our energies with the new variables:

$$E_D(X_{12}) = Tr\left(X_{12}^{\top} W_1 \, X_{12}\right)$$

$$E_R(X_{12}, X_{21}, \mathrm{P}_{12}, P_{21}) = \|P_{21}X_{12} - X_2\|_{M_2}^2 + \|P_{12}X_{21} - X_1\|_{M_1}^2$$

$$E_Q(X_{12}, P_{12}) = \|P_{12}X_2 - X_{12}\|_{M_1}^2$$

Optimization

We optimize the energy:

$$E(X_{12}, X_{21}, \mathrm{P}_{12}, P_{21}) = \alpha E_D(X_{12}) + \alpha E_D(X_{21}) +$$ Dirichlet

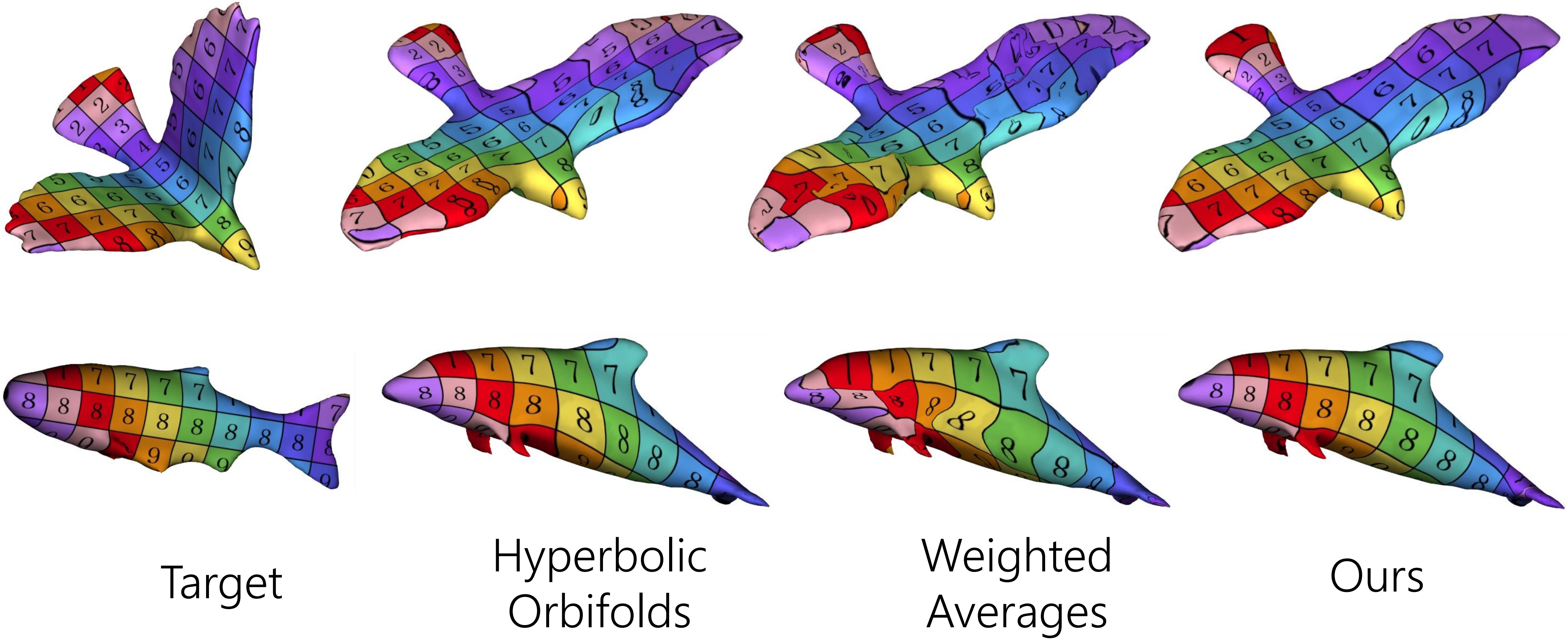
$$+ (1-\alpha) E_R(X_{12}, X_{21}, \mathrm{P}_{12}, P_{21}) +$$ Reversibility

$$+ \beta E_Q(X_{12}, P_{12}) + \beta E_Q(X_{21}, P_{21})$$ Penalty

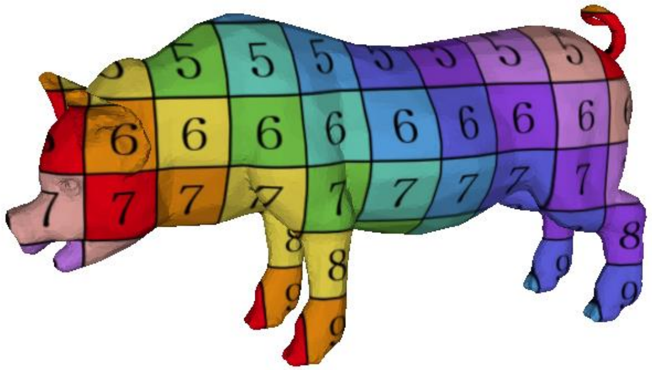
by alternatingly optimizing for each variable

- Optimize P_{12} or P_{21} using projection
- Optimize X_{12} or X_{21} by solving a linear system

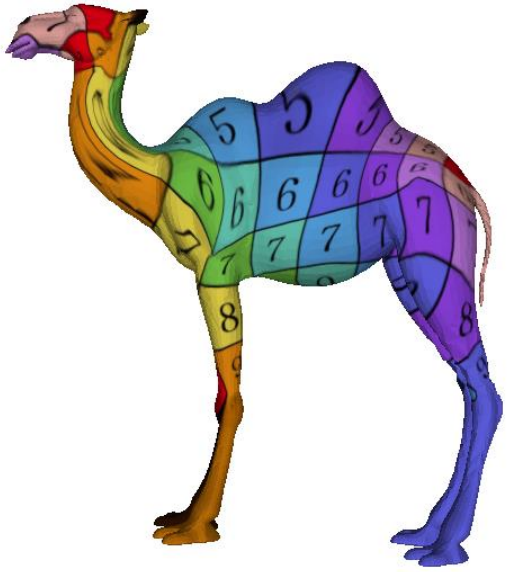
Results – SHREC'07



Results – SHREC'07

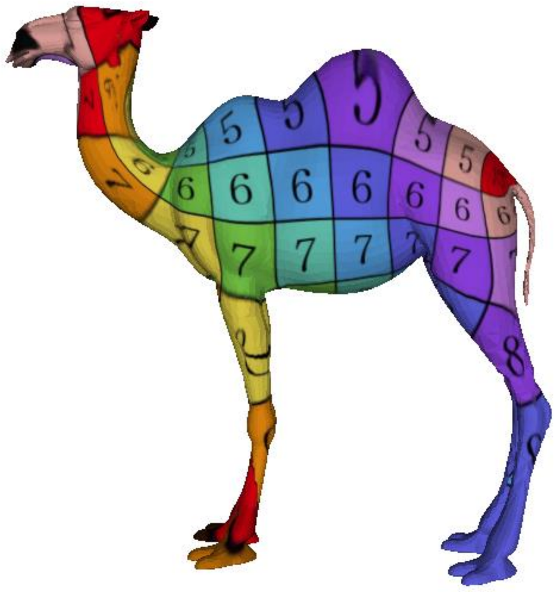


Target



Hyperbolic Orbifolds

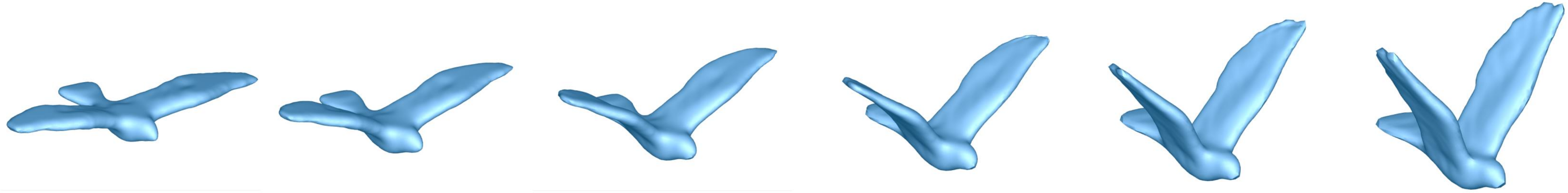
Weighted Averages



Ours

Results

Interpolation methods require the source and target shape to have a matching triangulation, that can be computed by our method



Heeren, Behrend, Martin Rumpf, Peter Schröder, Max Wardetzky and Benedikt Wirth, “Exploring the Geometry of the Space of Shells”, 2014