Homework 3: Calculus on Manifolds

Due April 10, 2019

Problem 1 (15 points)**.** In class, we have shown that the Laplacian is self-adjoint linear operator. In this problem, you will prove properties of its eigendecomposition. We will do this more generally for any self-adjoint operator A.

You will need the following definition: A Hermitian inner product $\langle \cdot, \cdot \rangle$ is a complex valued bilinear form with the following properties:

- $\langle u + v, w \rangle = \langle u, w \rangle + \langle v, w \rangle$
- $\langle u, v + w \rangle = \langle u, v \rangle + \langle u, w \rangle$
- $\langle \alpha u, v \rangle = \alpha \langle u, v \rangle$
- $\langle u, \alpha v \rangle = \bar{\alpha} \langle u, v \rangle$
- $\langle u, v \rangle = \overline{\langle v, u \rangle}$
- $\langle u, u \rangle \geq 0$

Let A be a self-adjoint linear operator. Recall that self-adjoint in this context means $\langle Ax, y \rangle = \langle x, Ay \rangle$. Prove that:

(a) All the eigenvalues of A are real.
(b) Any two eigenvectors e_i, e_j of A with distinct eigenvalues λ_i, λ_j are orthogonal, i.e. $\langle e_i, e_j \rangle = 0$ assuming $\lambda_i \neq \lambda_j$.

> For the coding assignments, we've provided you with helper functions for loading and plotting triangle meshes. Before starting the homework, take a look at `utils/` for `MATLAB` code or `utils.jl` for `Julia` code to familiarize yourself with the syntax.

Problem 2 (20 points)**.** The file `swissroll.txt` contains a point cloud in 3D with a two-dimensional structure. Implement any *two* of the methods we have discussed in class for embedding this data set into $\mathbb{R}^3$ and plot the resulting embeddings. Compare the results from the two embeddings. Starter code is in `embeddings.m` (`embeddings.jl`).

Problem 3 (40 points)**.** In this problem, you will implement the heat kernel signature for points on a surface. The reference paper for this method is **A Concise and Provably Informative Multi-Scale Signature Based on Heat Diffusion** (Sun, Ovsjanikov and Guibas; SGP 2009). See starter code in `hks.m` (`hks.jl`).

(a) Complete the function `laplacianSpectrum` for approximating the smallest k eigenvalues of the Laplacian of a triangulated surface, as well as their corresponding eigenfunctions. Don't forget to use a mass matrix!
(b) The heat kernel signature is a function $t > 0$ at each point x on a surface (eqn. (4) in the paper):

$$\mathrm{HKS}(x, t) := \sum_{i=0}^{\infty} e^{-\lambda_i t} \varphi_i(x)^2$$

where φ_i is the i-th Laplace-Beltrami eigenfunction normalized to square-integrate to 1. Implement an approximation of this function per-vertex in the function `HKS`.

(c) The code in `hks.m` (`hks.jl`) plots the HKS for the two hands and two feet of the human model. Play around with the number of eigenfunctions and number of samples used to approximate the HKS. Discuss the pattern you see. Is it what you expect?

(d) For a point $x_0 \in S$, plot the function $\|\text{HKS}(x_0) - \text{HKS}(x)\|_2$ as a function of $x \in S$, where x_0 is a vertex on the human hand model. Is the HKS discriminative?

Problem 4 (Extra credit: 20 points)**.** In class we gave a short argument for the divergence theorem on a surface. For extra credit, formalize the proof.