Representing Surfaces

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Today’s Plan

Step up one dimension from curves to surfaces.

- Theoretical definition
- Discrete representations
- Higher dimensionality
Briefly Will Mention

Step up $n$ dimensions from surfaces to (sub)manifolds.

Easier transition.

*Not entirely true: e.g. topology of 3-manifolds*
Our Focus

$\subseteq \mathbb{R}^3$

Embedded geometry

http://web.mit.edu/manoli/crust/www/slides/piggy.jpg
Q: What is an embedded surface?
Pathological Cases

\[ f(u, v) = (u, u^2, \cos u) \]

\[ f(u, v) = (0, 0, 0) \]

\[ f(u, v) = (u, v^3, v^2) \]

What condition do we need to add?
Differential of a Function

\[ f : \mathbb{R}^n \to \mathbb{R}^m \]

Matrix:

\[ Df = \left( \frac{\partial f_i}{\partial x_j} \right) \in \mathbb{R}^{m \times n} \]

Linear operator:

\[ df_p : T_p \mathbb{R}^n \to T_{f(p)} \mathbb{R}^m \]
Injective/Regular/One-to-One

\[ f : \mathbb{R}^n \rightarrow \mathbb{R}^m \]

Matrix:

\[ Df \in \mathbb{R}^{m \times n} \text{ full rank} \]

Linear operator:

\[ df_p : T_p \mathbb{R}^n \rightarrow T_{f(p)} \mathbb{R}^m \text{ full rank} \]
Next Issue

One function isn’t enough!

Major difference from curves!

\[ f : \mathbb{R}^2 \rightarrow \mathbb{R}^3 \]
Recall: Differential Geometry Definition

\[ \gamma_p : (a, b) \rightarrow C \cap U \]
A surface is a set of points with certain properties. It is not a function.
Theoretical Definition of Surface

\[ \subseteq \mathbb{R}^3 \]

\[ \subseteq \mathbb{R}^2 \]
**Theoretical Definition: Manifold**

**Definition** (Submanifold of \( \mathbb{R}^n \), with and without boundary). A set \( \mathcal{M} \subseteq \mathbb{R}^n \) is an \( m \)-dimensional submanifold of \( \mathbb{R}^n \) if for each \( p \in \mathcal{M} \) there exist open sets \( U \subseteq \mathbb{R}^m, W \subseteq \mathbb{R}^n \) and a function \( g : U \cap \mathcal{H}_m \to \mathcal{M} \cap W \) such that \( p \in W \) and \( g \) is a one-to-one and smooth map whose Jacobian is rank-\( m \) and admitting a continuous inverse \( g^{-1} : W \cap \mathcal{M} \to U \).
A surface is locally planar.
Tangent Space

\[ T_p \mathcal{M} = \gamma'(0), \text{ where } \gamma(0) = p \]
= image\((dg_p)\)

\[ \mathcal{M} \subseteq \mathbb{R}^n \]

\[ U \subseteq \mathbb{R}^m \]
Normal Space

\[ N_p \mathcal{M} := (T_p \mathcal{M})^\perp \]

\[ \mathcal{M} \subseteq \mathbb{R}^n \]

\[ U \subseteq \mathbb{R}^m \]
Orientable Submanifold

Admits a continuous map

\[ n(p) : M \setminus \partial M \rightarrow S^{n-1} \]

with

\[ n(p) \in N_p M \]

Orientable

Not Orientable
Definition 4.2 (Manifold). An $m$-dimensional (topological) manifold $M$ is a Hausdorff space for which each $p \in M$ admits open sets $U \subseteq \mathbb{R}^m$, $W \subseteq M$ and a homeomorphism (continuous map with continuous inverse) $g : U \to W$.

To think about:
No notion of normal!
Tangent vectors exist but have no length!
How do you detect orientability?
What is a discrete surface?
How do you store it?
Common Representation

Triangle mesh

http://graphics.stanford.edu/data/3Dscanrep/stanford-bunny-cebal-ssh.jpg
http://www.stat.washington.edu/wxs/images/BUNMID.gif
After Some Cleaning

\[ M = (V, T) \]

What conditions are needed?

Triangle mesh
What is a Discrete Surface?
To read: More general story

“Orientable combinatorial manifold”
Dimensionality Structure

- Face: Dimension 2
- Edge: Dimension 1
- Vertex: Dimension 0

Simplicial complex
Nonmanifold Edge
1. Each **edge** is incident to one or two faces

2. **Faces** incident to a vertex form a closed or open fan
1. Each edge is incident to one or two faces

2. Faces incident to a vertex form a closed or open fan

Assume meshes are manifold (for now)
Piecewise linear faces are reasonable building blocks.
Additional Advantages

- Simple to render
- Arbitrary topology possible
- Basis for subdivision, refinement
Easy-to-Violate Assumption

“Triangle soup”
Invalid Meshes vs. Bad Meshes

Nonuniform areas and angles
Why is Meshing an Issue?

How to you interpret one value per vertex?

Topology [tuh-pol-uh-jee]: The study of geometric properties that remain invariant under certain transformations
Mesh Topology vs. Geometry

Geometry: “This vertex is at \((x,y,z)\).”
Mesh Topology vs. Geometry

Topology:
“These vertices are connected.”
Triangle Mesh

\[ V = (v_1, v_2, \ldots, v_n) \subseteq \mathbb{R}^n \]
\[ E = (e_1, e_2, \ldots, e_k) \subseteq V \times V \]
\[ F = (f_1, f_2, \ldots, f_m) \subseteq V \times V \times V \]

Plus manifold conditions
Valence = 6

Synonym: Degree
Euler Characteristic

\[ V - E + F := \chi \]

\[ \chi = 2 - 2g \]

- \( g = 0 \)
- \( g = 1 \)
- \( g = 2 \)
Closed mesh: Easy estimates!

Each edge is adjacent to two faces. Each face has three edges.

\[ V - E + F := \chi \]

\[ 2E = 3F \]
Consequences for Triangle Meshes

\[ V - \frac{1}{2} F := \chi \]

“Each edge is adjacent to two faces. Each face has three edges.”

\[ 2E = 3F \]

Closed mesh: Easy estimates!
Consequences for Triangle Meshes

Big number: \( V - \frac{1}{2} F \) := \( \chi \)

Small number: \( F \approx 2V \)

Closed mesh: Easy estimates!
Consequences for Triangle Meshes

\[ E \approx 3V \]
\[ F \approx 2V \]

average valence \( \approx 6 \)

Why?!
Orientability
Smooth Surface Definition

Continuous field of normal vectors
Issue on Triangle Mesh

What happens on edges/vertices?
Right-Hand Rule

http://viz.aset.psu.edu/gho/sem_notes/3d_fundamentals/html/3d_coordinates.html
http://mathinsight.org/stokes_theorem_orientation
Normal field isn’t continuous
Data Structures for Surfaces

Must represent geometry and topology.
Simplest Format

```
x1 y1 z1 / x2 y2 z2 / x3 y3 z3
x1 y1 z1 / x2 y2 z2 / x3 y3 z3
x1 y1 z1 / x2 y2 z2 / x3 y3 z3
x1 y1 z1 / x2 y2 z2 / x3 y3 z3
x1 y1 z1 / x2 y2 z2 / x3 y3 z3
x1 y1 z1 / x2 y2 z2 / x3 y3 z3
```

No topology!

```
glBegin(GL_TRIANGLES)
```

Triangle soup
Factor Out Vertices

\[
\begin{align*}
&f 1 5 3 \\
&f 5 1 2 \\
&\ldots \\
&v 0.2 1.5 3.2 \\
&v 5.2 4.1 8.9 \\
&\ldots 
\end{align*}
\]

.obj format

Shared vertex structure
for \ i=1 \ to \ n \\
for \ each \ vertex \ v \\
v = 0.5*v + 0.5*(average \ of \ neighbors);
Typical Queries

- Neighboring vertices to a vertex
- Neighboring faces to an edge
- Edges adjacent to a face
- Edges adjacent to a vertex
- ...

Mostly localized
Typical Queries

- Neighboring vertices to a vertex
- Neighboring faces to an edge
- Edges adjacent to a face
- Edges adjacent to a vertex
- ...

Mostly localized
Pieces of Halfedge Data Structure

- Vertices
- Faces
- Half-edges

Structure tuned for meshes
Halfedge?

Associated with single face!

Oriented edge
Halfedge Data Types

Vertex stores:
• Arbitrary outgoing halfedge
Halfedge Data Types

Face stores:
• Arbitrary adjacent halfedge
Halfedge Data Types

Halfedge stores:
• Flip
• Next
• Face
• Vertex
Iterate(v):
startEdge = v.out;
e = startEdge;
do
      process(e.flip.from)
      e = e.flip.next
while e != startEdge
Streaming Compression of Triangle Meshes

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SQuad: Compact Representation for Triangle Meshes

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Scalar Functions

Map points to real numbers
Discrete Scalar Functions

Map vertices to real numbers
What is the integral of $f$?

\[ \int_{M} f \, dA \]
Discrete version of $dA$
Dual Complex

Valence 3
One Surface, Two Meshes

http://www.grasshopper3d.com/group/kangaroo/forum/topics/isosurface-dynamic-remeshing
One Surface, Two Halfedges
Missing Operation
Rotation Operation

\[ e \rightarrow \text{Rot} \rightarrow \text{Rot} = e \rightarrow \text{Flip} \]
Topological Operations

Original Mesh Segment

Vertex Removal
Edge Collapse
Face Collapse

Necessary bookkeeping for each operation?
Complex data structures enable simpler traversal at cost of more bookkeeping.
Not the Only Model

Implicit surfaces

Not the Only Model

Smoothed-particle hydrodynamics

http://www.itsartmag.com/features/cgfluids/
https://developer.nvidia.com/content/fluid-simulation-alice-madness-returns
AN IMPLICIT SURFACE TENSION MODEL

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ABSTRACT

A new implicit model for surface tension at a two-fluid interface is proposed for use in computational models of flows with free surfaces and its performance is compared to an existing explicit model. The new model is based on an evolution equation for surface curvature that includes the influence of advection as well as surface tension. A detailed development of the new model is presented as are the details of the computational implementation. The performance of the new model is compared to an existing explicit model by using both models to predict the surface dynamics of several two-dimensional configurations. It is concluded that the new implicit surface tension model does perform better for configurations with a large surface tension coefficient. It is shown that, for several cases, the time step size is no longer limited by surface tension stability considerations (as it was using the explicit model), but rather by other limitations inherent in the existing volume advection algorithm.

INTRODUCTION

Incompressible flows with a free-surface exist in many industrial applications. Some examples include fuel atomization in internal combustion engines, droplet size control in ink-jet printers, formation of lead shot, control of liquid spacecraft propellant in low gravity, and the spinning of synthetic fibers. The technology for some of these applications has been developed by heavily relying on experimental study of the specific process involved. For others, such as spacecraft propellant management, experimental studies are prohibitively expensive and the ability to computationally model these processes is essential for their development.

The modeling of flows with a free surface presents challenges unlike other types of flow problems in that a boundary condition must be applied at the free surface which is often in a transient state and irregularly shaped. This problem is exacerbated when the force due to surface tension...
Obtaining Geometry

May have to convert to mesh

Cleanest: Design software
Obtaining Geometry

Catmull-Clark

Cleanest: Design software

Halfedge suited for subdivision!
Obtaining Geometry

Volumetric extraction
Obtaining Geometry

Marching cubes: Isosurface extraction

Volumetric extraction

http://en.wikipedia.org/wiki/Marching_cubes
Obtaining Geometry

Point clouds

Well-behaved dual mesh

Delaunay Triangulation

Strategies for Surface Delaunay

- **Tangent plane**
  Derive local triangulation from tangent projection

- **Restricted Delaunay**
  Usual Delaunay strategy but in smaller part of $\mathbb{R}^3$

- **Inside/outside labeling**
  Find inside/outside labels for tetrahedra

- **Empty balls**
  Require existence of sphere around triangle with no other point

*Delaunay Triangulation Based Surface Reconstruction: Ideas and Algorithms*  
Cazals and Giesen 2004
Poisson Surface Reconstruction
Kazhdan, Bolitho and Hoppe (SGP 2006)
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