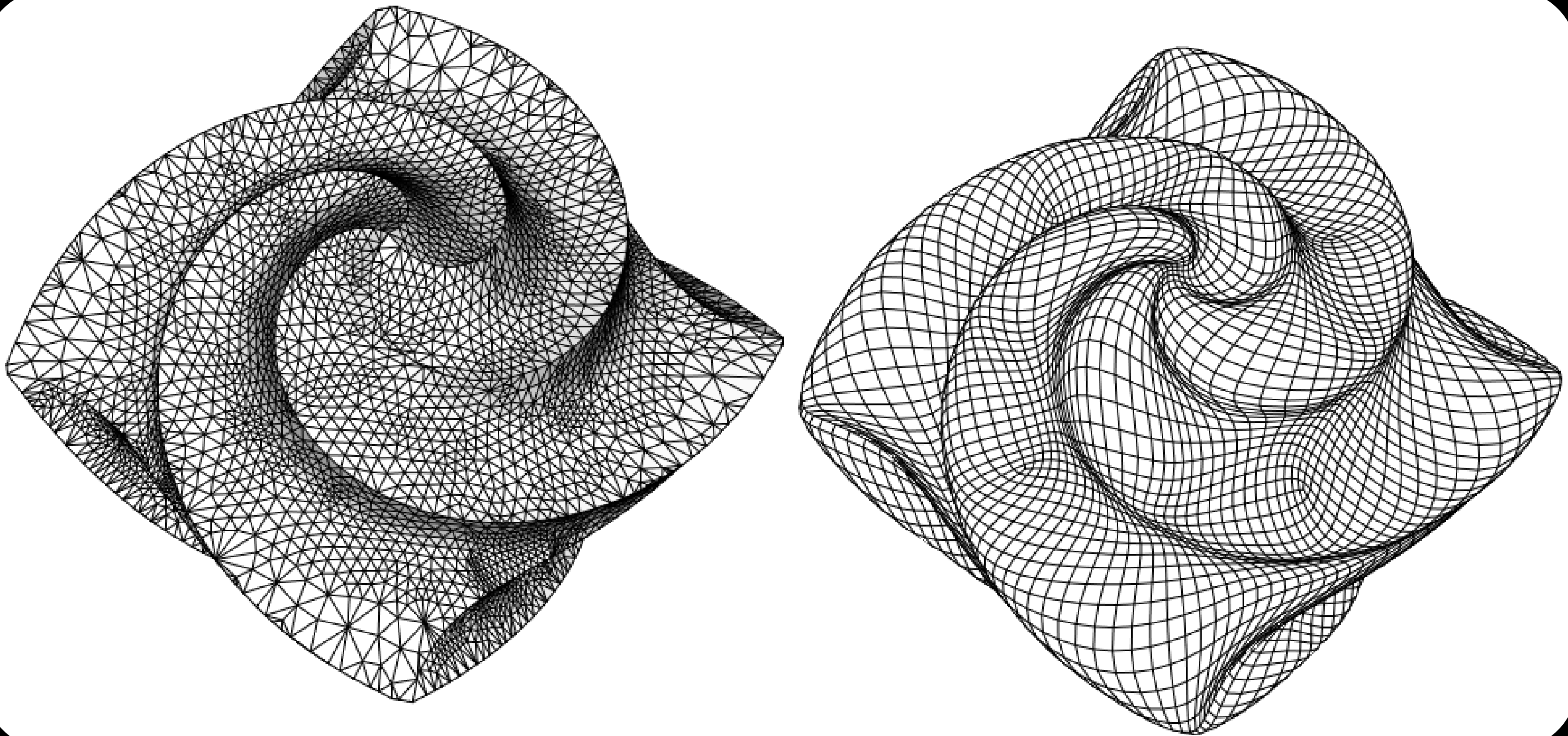
http://alice.loria.fr/index.php/publications.html?redirect=0&Paper=VSDM@2011&Author=levy



Representing Surfaces

Justin Solomon

MIT, Spring 2019

Today's Plan

Step up

one dimension

from curves to surfaces.

- Theoretical definition
- Discrete representations
- Higher dimensionality

Briefly Will Mention

Step up

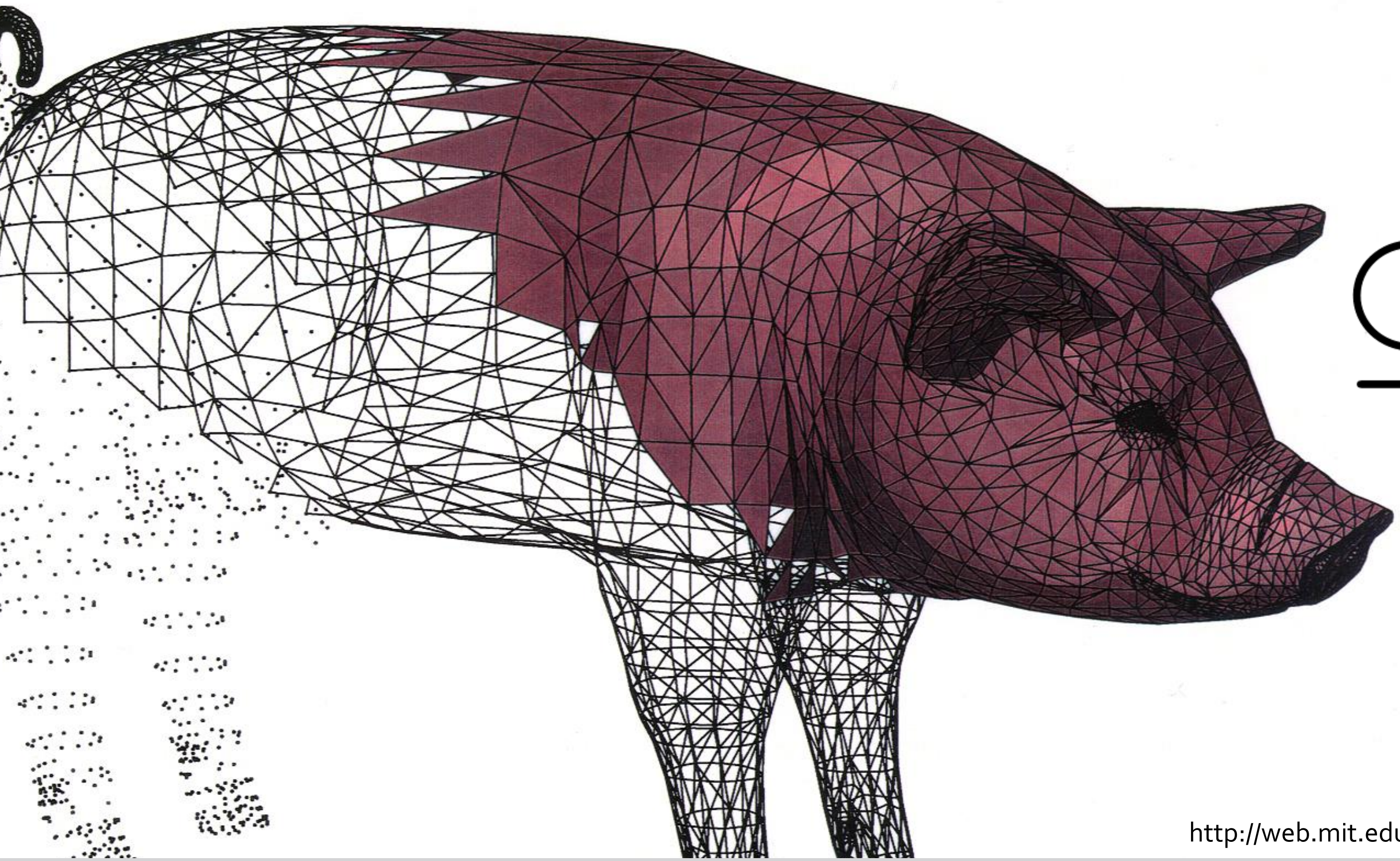
***n* dimensions**

from surfaces to (sub)manifolds.

Easier transition.

Not entirely true:
e.g. topology of 3-manifolds

Our Focus



$$\subseteq \mathbb{R}^3$$

http://web.mit.edu/manoli/crust/www/slides/piggy.jpg

Embedded geometry

Q:

What is an
embedded surface?

Parametric Surface

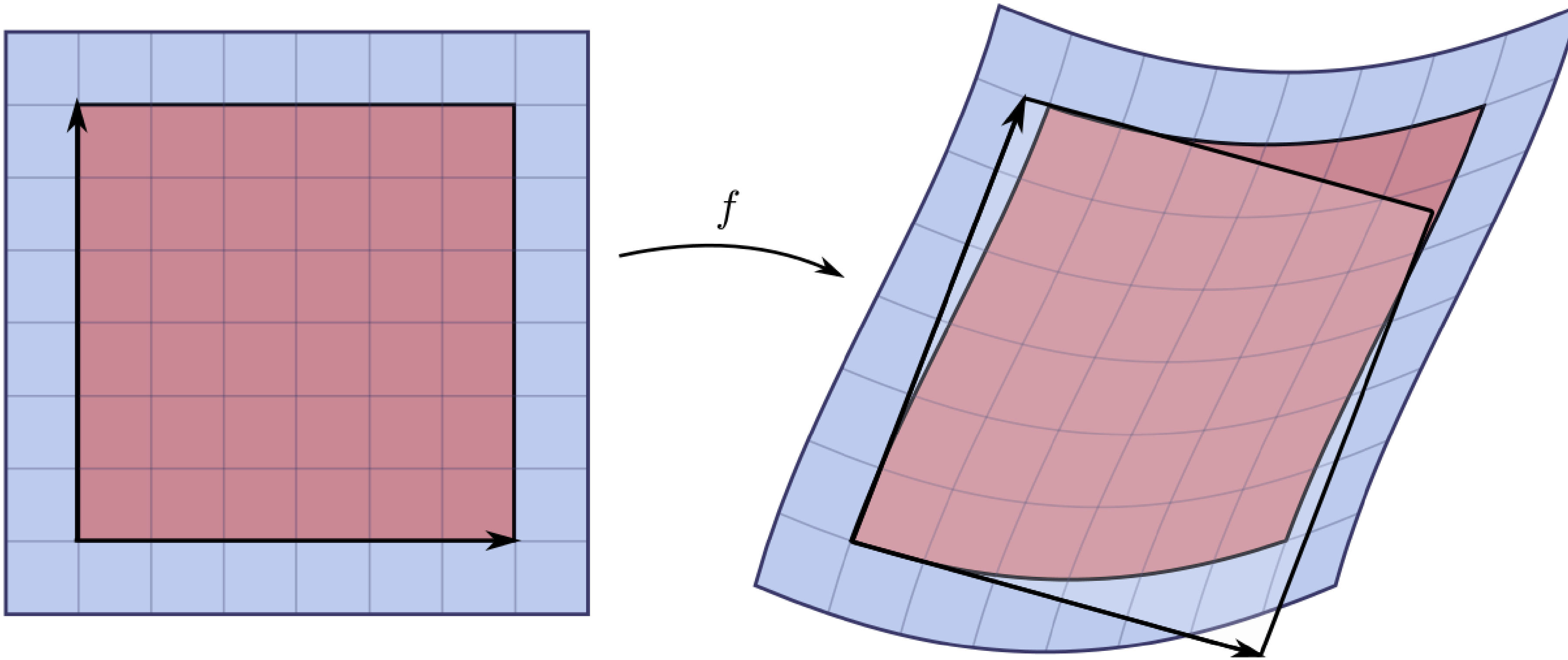


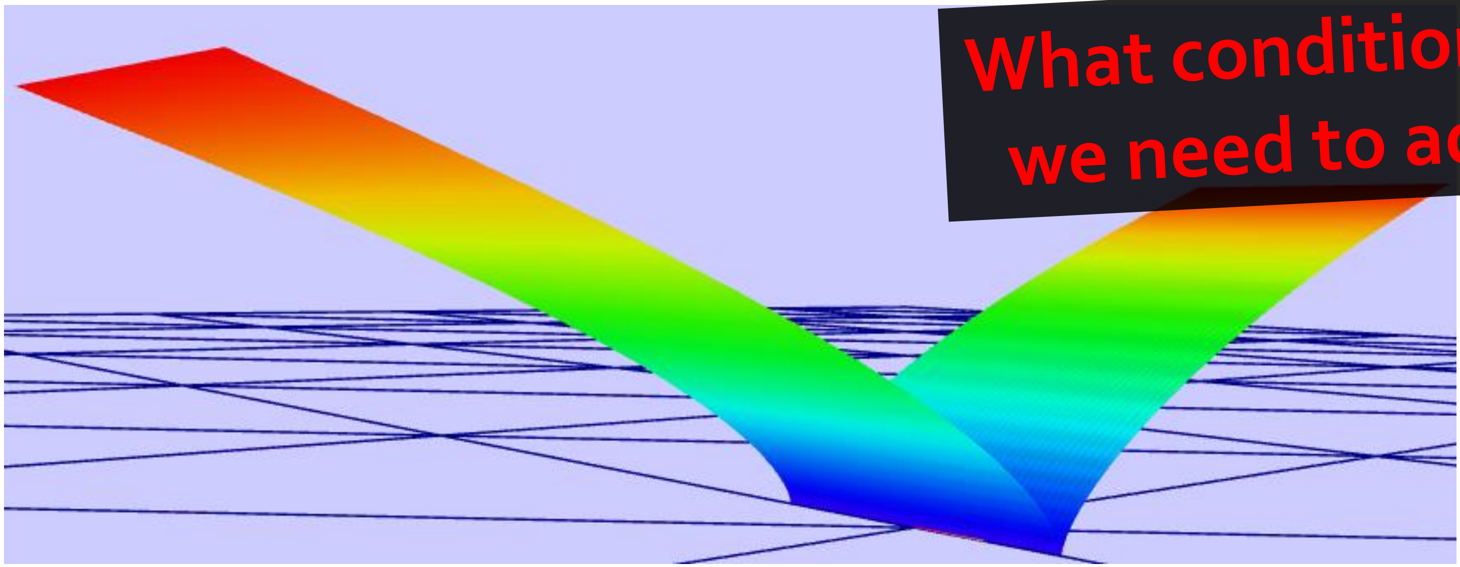
Image from Wikipedia

Pathological Cases

$$f(u,v) = (u, u^2, \cos u)$$

$$f(u,v) = (0,0,0)$$

$$f(u,v) = (u, v^3, v^2)$$



What condition do we need to add?

Differential of a Function

$$f : \mathbb{R}^n \to \mathbb{R}^m$$

Matrix:

$$Df = \left(\frac{\partial f_i}{\partial x_j}\right) \in \mathbb{R}^{m \times n}$$

Linear operator:

$$df_{\mathbf{p}} : T_{\mathbf{p}}\mathbb{R}^n \to T_{f(\mathbf{p})}\mathbb{R}^m$$

Injective/Regular/One-to-One

$$f : \mathbb{R}^n \to \mathbb{R}^m$$

Matrix:

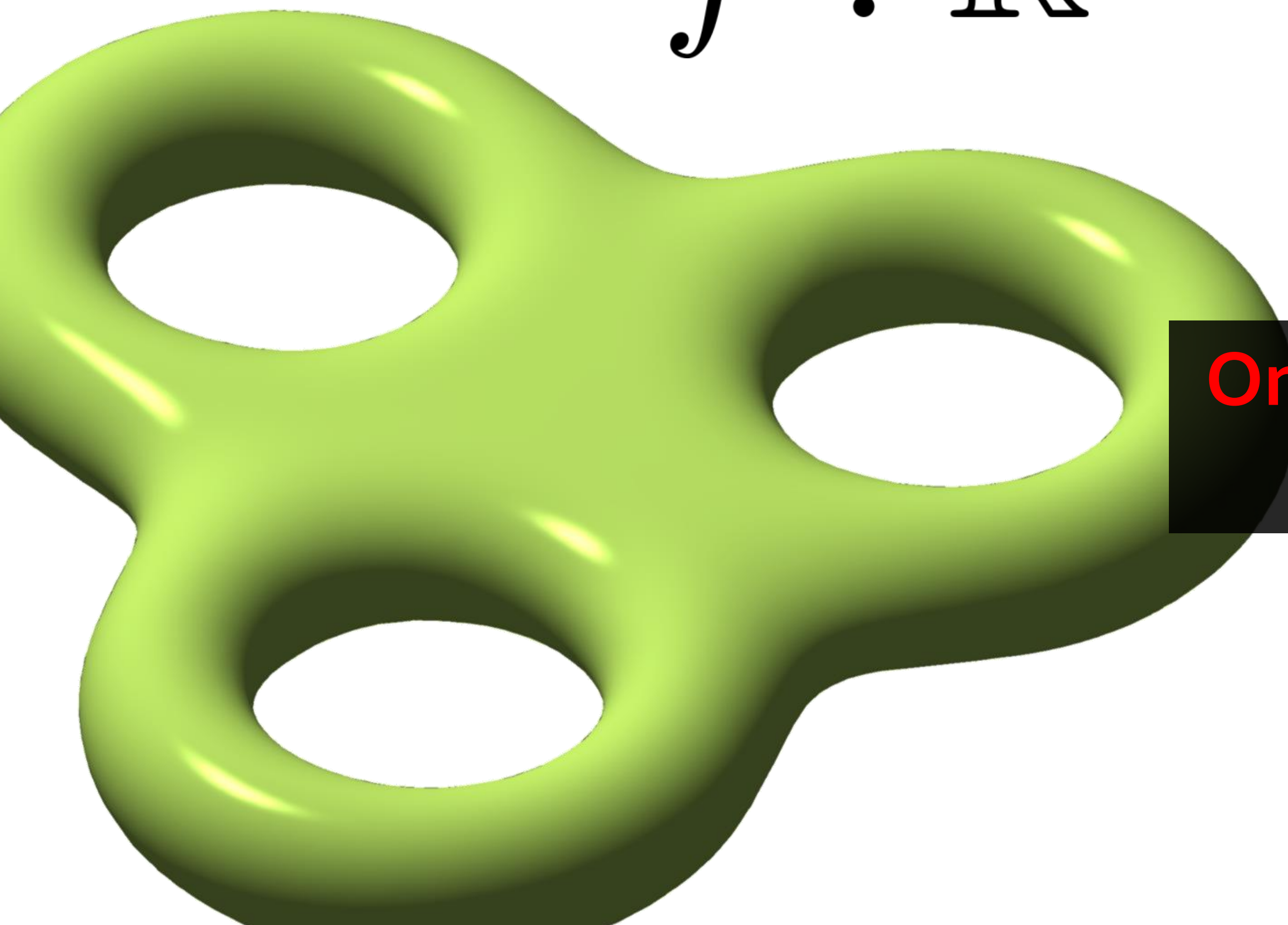
$$Df \in \mathbb{R}^{m \times n} \text{ full rank}$$

Linear operator:

$$df_{\mathbf{p}} : T_{\mathbf{p}}\mathbb{R}^n \to T_{f(\mathbf{p})}\mathbb{R}^m \text{ full rank}$$

Next Issue

$$f : \mathbb{R}^2 \rightarrow \mathbb{R}^3 \ ?$$



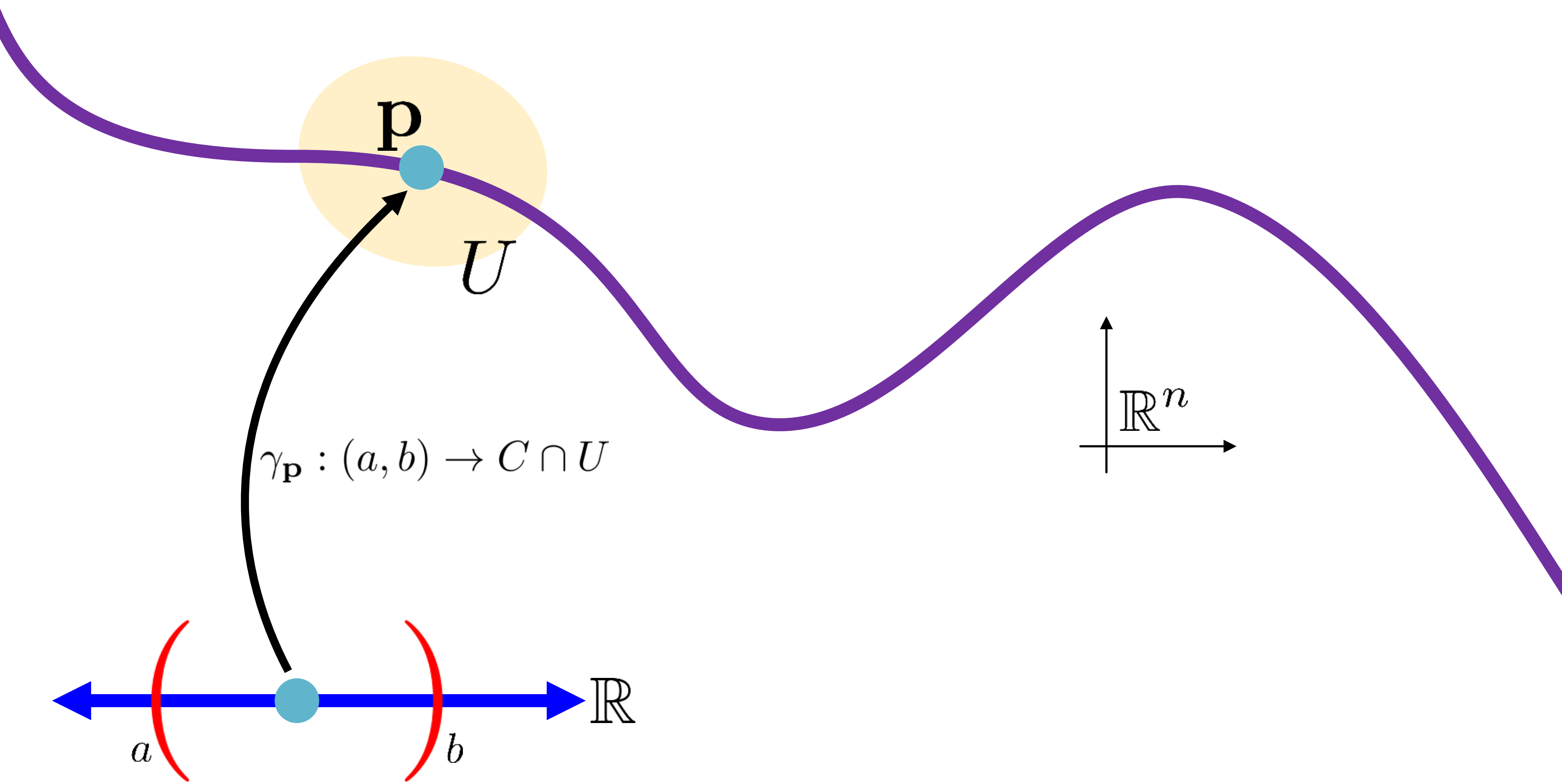
One function isn't enough!

Major difference from curves!

https://en.wikipedia.org/wiki/Triple_torus

Recall:

Differential Geometry Definition



Just Like Curves

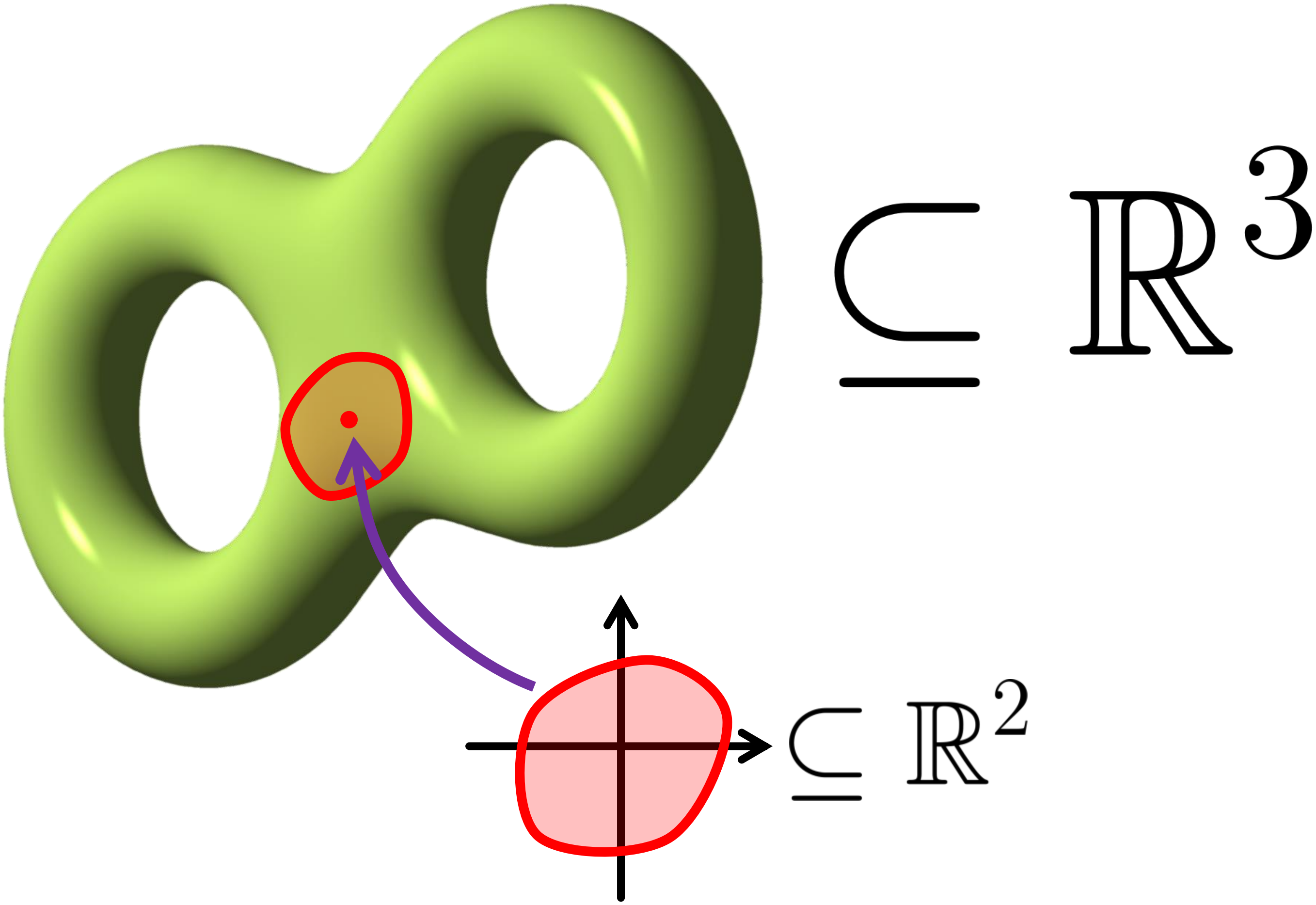
A surface is a

set of points

with certain properties.

It is not a function.

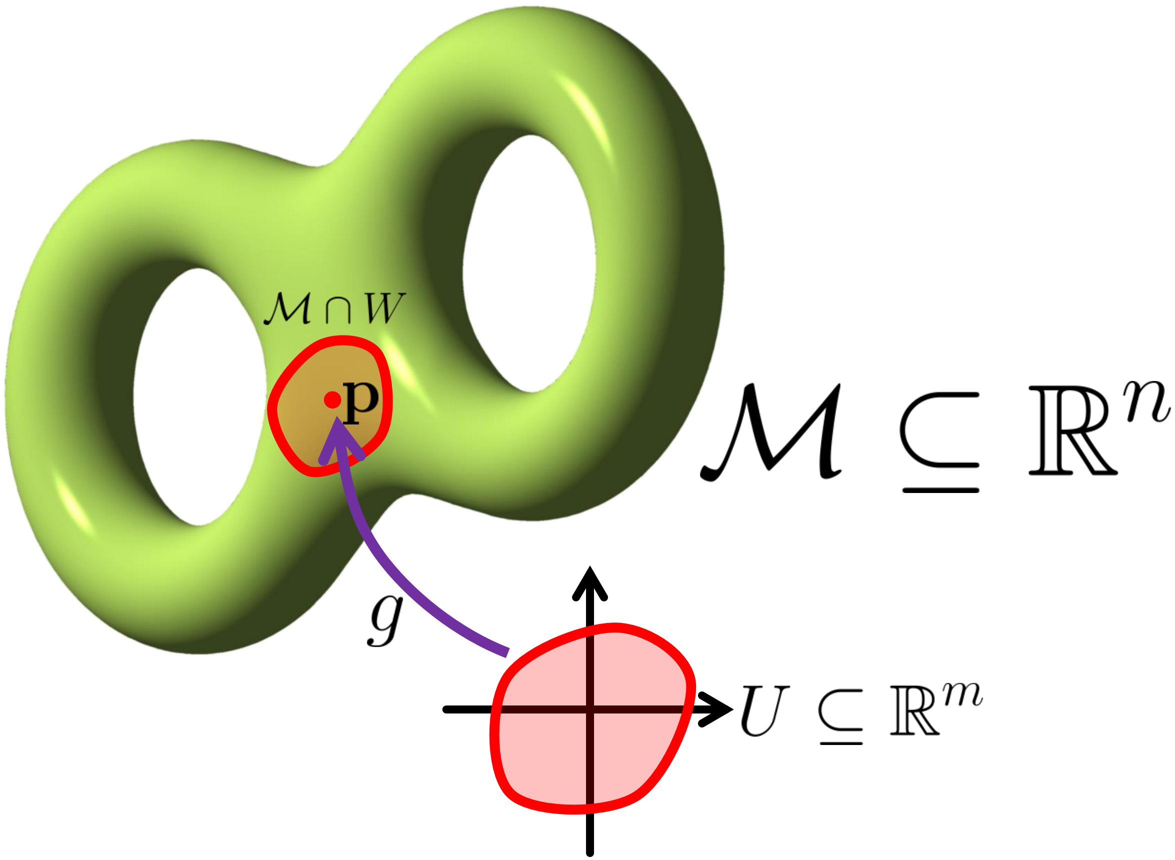
Theoretical Definition of Surface



http://upload.wikimedia.org/wikipedia/commons/b/bc/Double_torus_illustration.png

Theoretical Definition: Manifold

Definition (Submanifold of $\mathbb{R}^n$, with and without boundary). *A set $\mathcal{M} \subseteq \mathbb{R}^n$ is an m-dimensional submanifold of $\mathbb{R}^n$ if for each $\mathbf{p} \in \mathcal{M}$ there exist open sets $U \subseteq \mathbb{R}^m$, $W \subseteq \mathbb{R}^n$ and a function $g : U \cap \mathcal{H}_m \to \mathcal{M} \cap W$ such that $\mathbf{p} \in W$ and g is a one-to-one and smooth map whose Jacobian is rank-m and admitting a continuous inverse $g^{-1} : W \cap M \to U$.*



Differential Geometer’s Mantra

A surface is

locally planar.

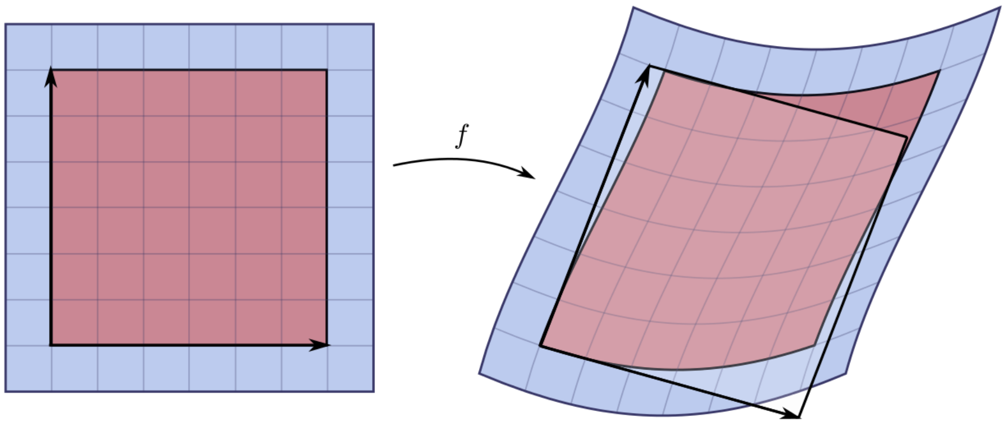
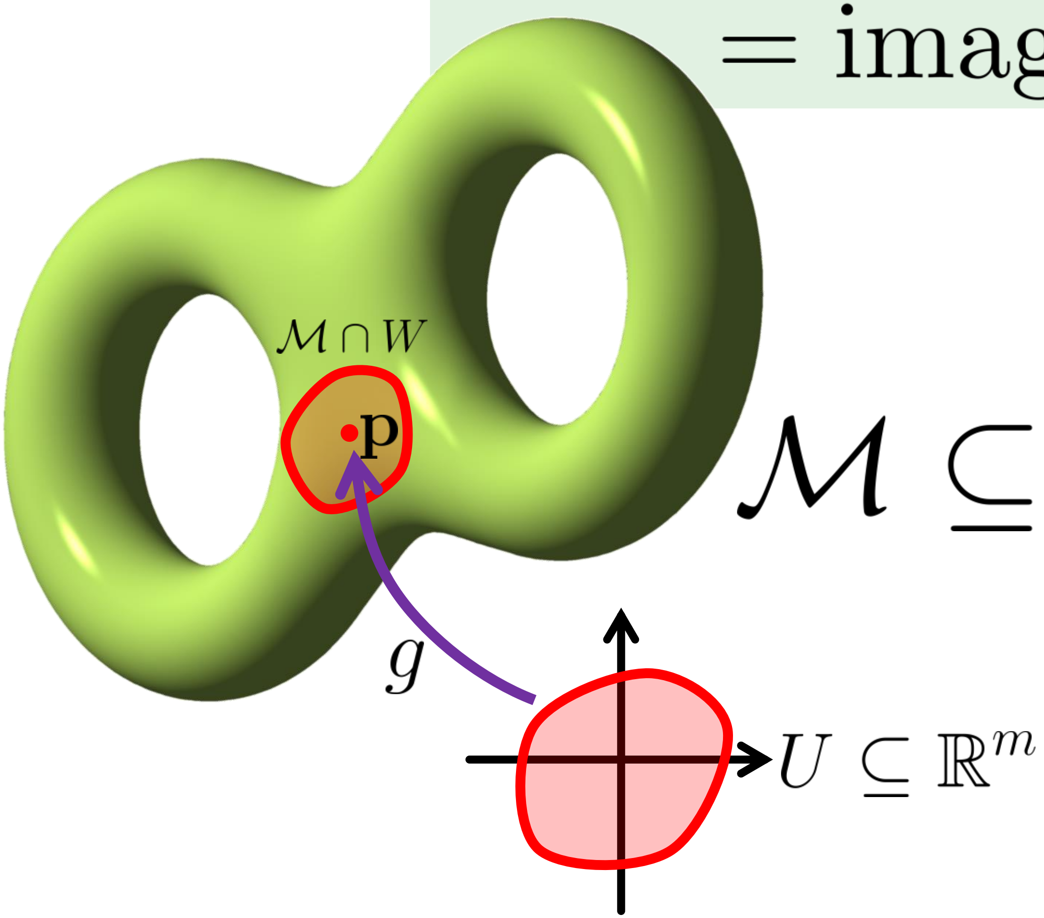


Image from Wikipedia

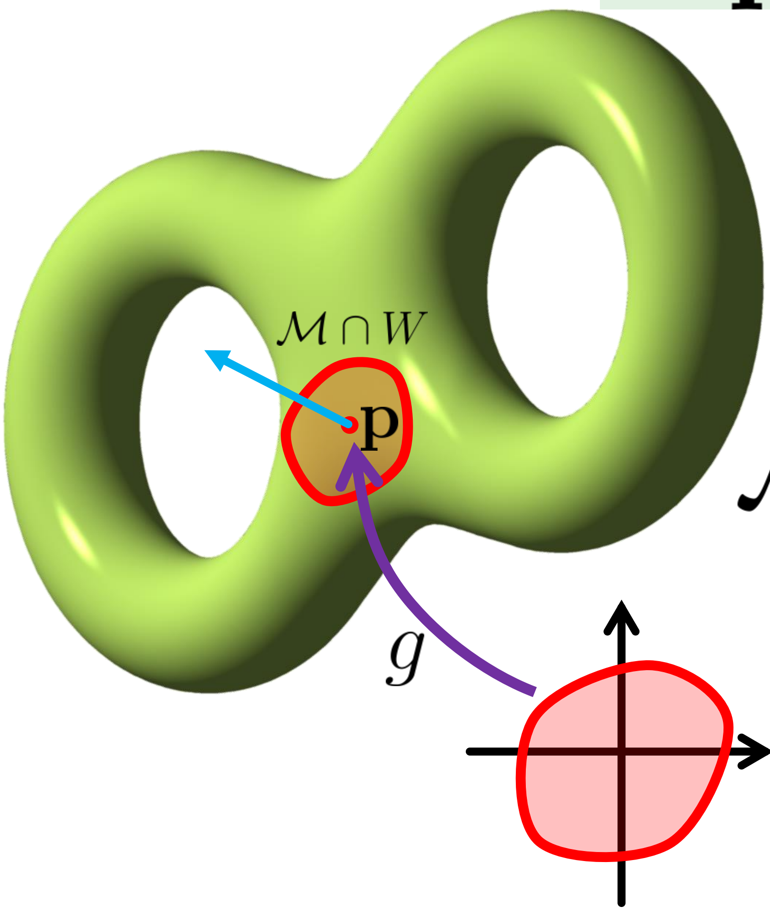
Tangent Space

$$T_{\mathbf{p}}\mathcal{M} = \gamma'(0), \text{ where } \gamma(0) = \mathbf{p}$$
$$= \text{image}(dg_{\mathbf{p}})$$



Normal Space

$$N_{\mathbf{p}}\mathcal{M} := (T_{\mathbf{p}}\mathcal{M})^{\perp}$$



$\mathcal{M} \subseteq \mathbb{R}^n$

$U \subseteq \mathbb{R}^m$

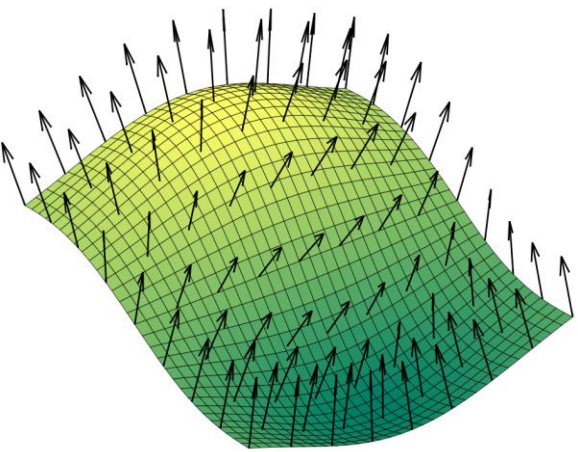
Orientable Submanifold

Admits a continuous map

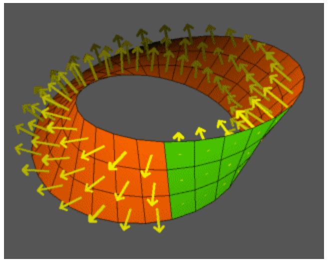
$$\mathbf{n}(\mathbf{p}) : \mathcal{M} \backslash \partial\mathcal{M} \to \mathcal{S}^{n-1}$$

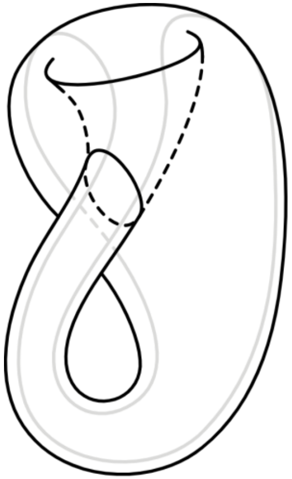
with

$$\mathbf{n}(\mathbf{p}) \in N_{\mathbf{p}}\mathcal{M}$$

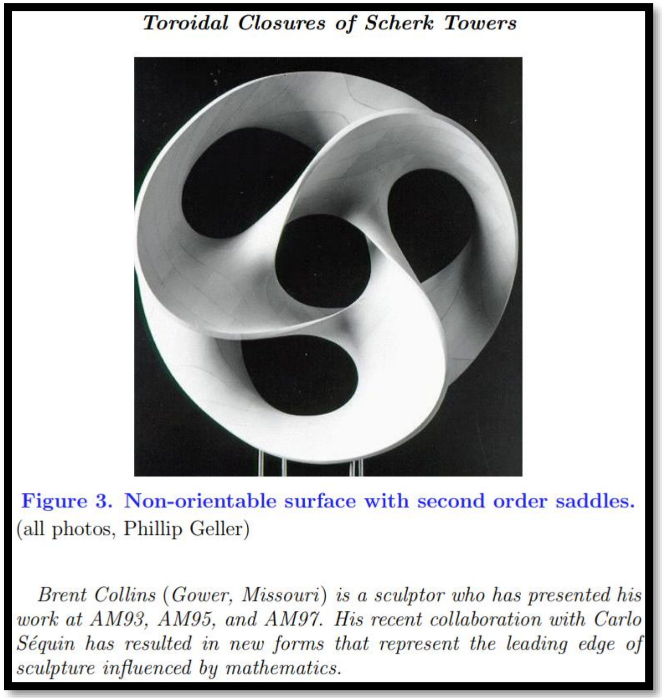


Orientable





Not Orientable



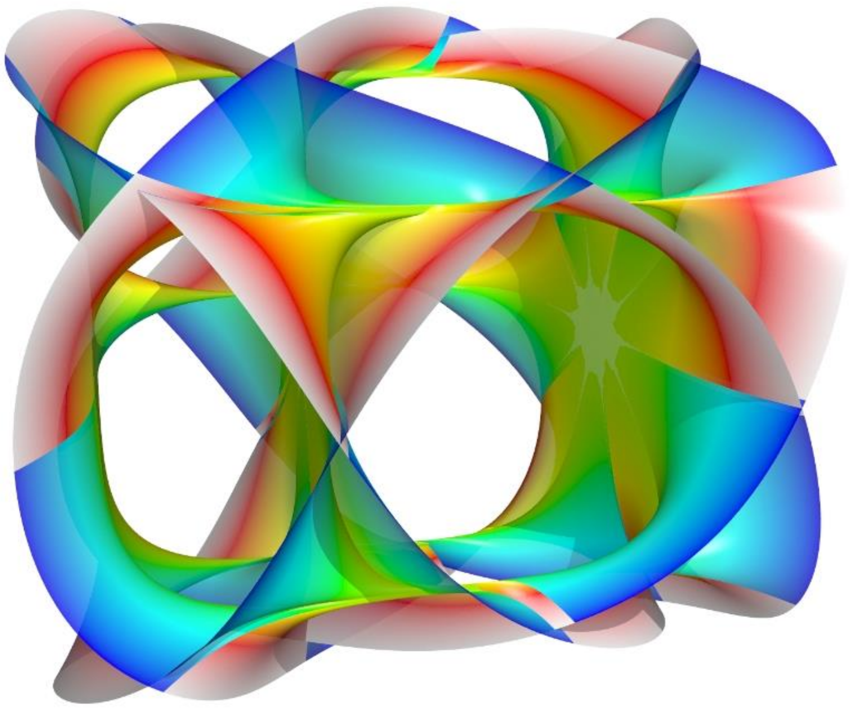
Toroidal Closures of Scherk Towers

Figure 3. Non-orientable surface with second order saddles.
(all photos, Phillip Geller)

Brent Collins (Gower, Missouri) is a sculptor who has presented his work at AM93, AM95, and AM97. His recent collaboration with Carlo Séquin has resulted in new forms that represent the leading edge of sculpture influenced by mathematics.

More General Definition: Manifold

Definition 4.2 (Manifold). *An m-dimensional (topological)* manifold $\mathcal{M}$ *is a Hausdorff space for which each* $\mathbf{p} \in \mathcal{M}$ *admits open sets* $U \subseteq \mathbb{R}^m, W \subseteq \mathcal{M}$ *and a homeomorphism (continuous map with continuous inverse)* $g : U \to W$.



To think about:

No notion of normal!

Tangent vectors exist but have no length!

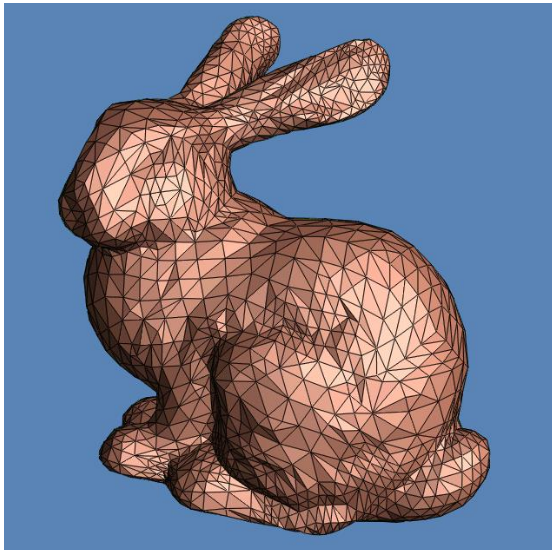
How do you detect orientability?

http://www.math.sjsu.edu/~simic/Pics/Calabi-Yau.jpg

No Euclidean embedding

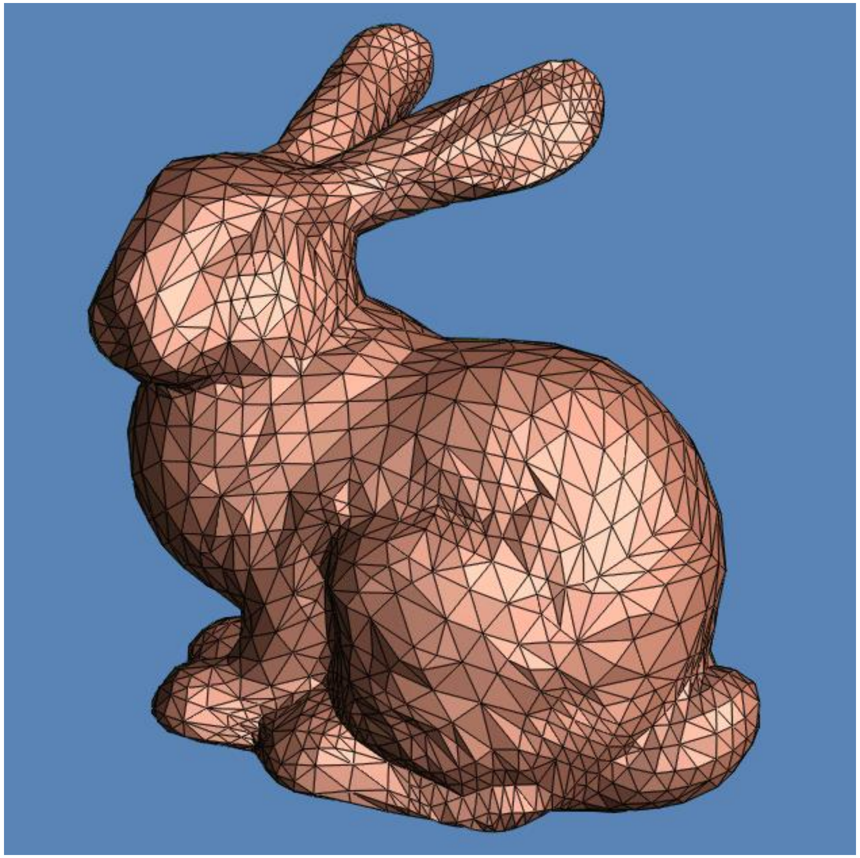
Discrete Problem

What is a discrete surface?
How do you store it?



http://graphics.stanford.edu/data/3Dscanrep/stanford-bunny-cebal-ssh.jpg
http://www.stat.washington.edu/wxs/images/BUNMID.gif

Common Representation



http://graphics.stanford.edu/data/3Dscanrep/stanford-bunny-cebal-ssh.jpg
http://www.stat.washington.edu/wxs/images/BUNMID.gif

Triangle mesh

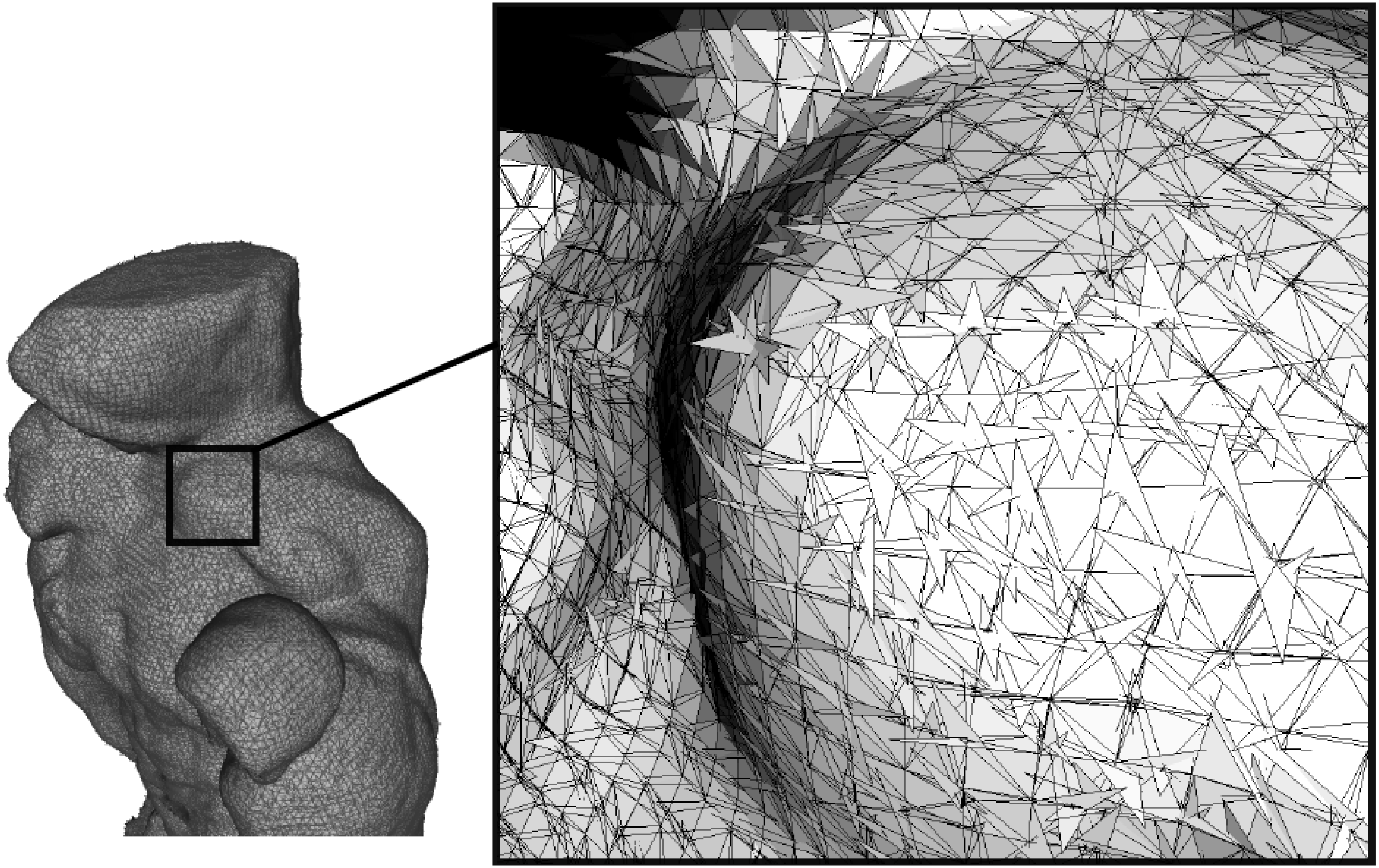
After Some Cleaning

$$M = (V, T)$$

What conditions are needed?

Triangle mesh

What is a Discrete Surface?



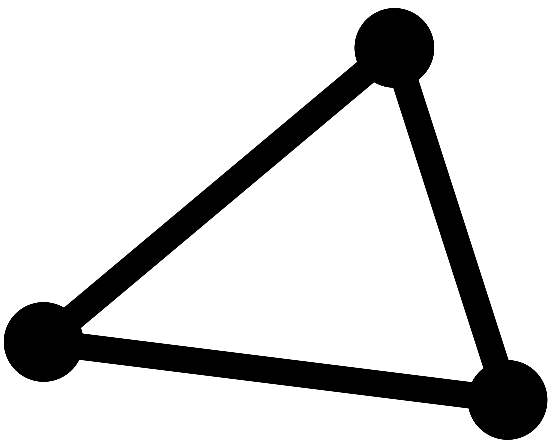
http://igl.ethz.ch/projects/parameterization/rangemap-param/rangemap-param.pdf

To read: More general story

“Orientable combinatorial manifold”

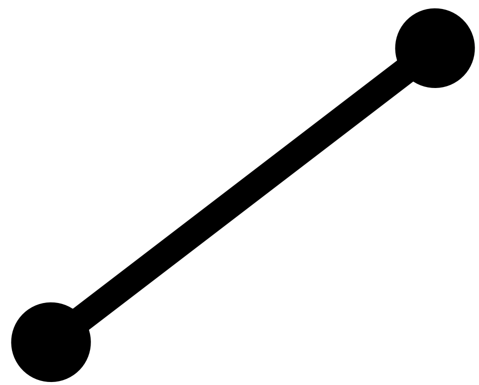
Dimensionality Structure

Simplicial complex



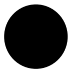
Face

Dimension 2



Edge

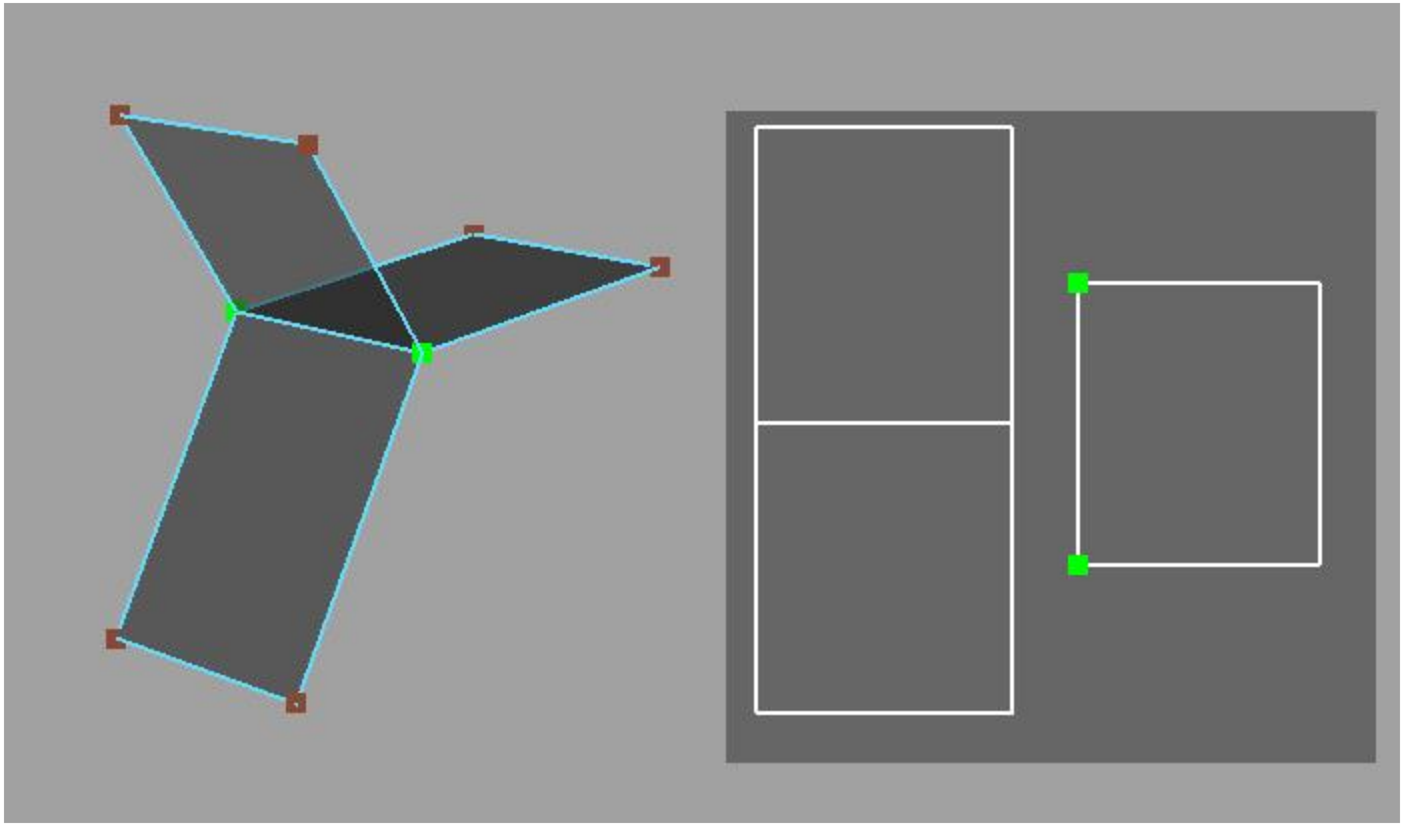
Dimension 1



Vertex

Dimension 0

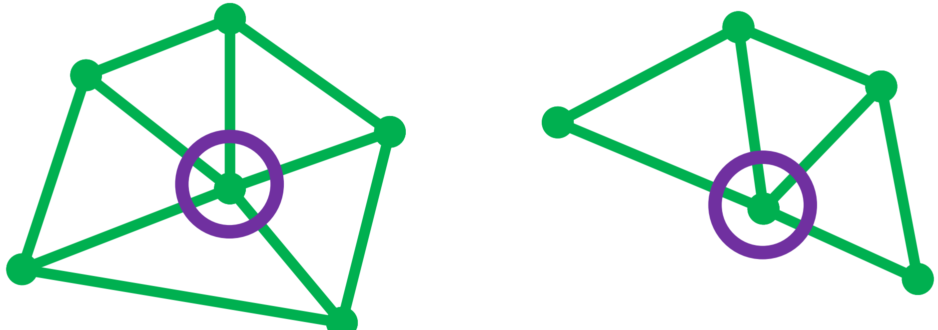
Nonmanifold Edge



http://blog.mixamo.com/wp-content/uploads/2011/01/nonmanifold.jpg

Manifold Mesh

1. Each edge is incident to one or two faces
2. Faces incident to a vertex form a closed or open fan



http://www.cs.mtu.edu/~shene/COURSES/cs3621/SLIDES/Mesh.pdf

Manifold Mesh

1. Each **edge** is incident to one or two faces
2. **Faces** incident to a vertex form a closed or open fan

Assume meshes are manifold (for now)

http://www.cs.mtu.edu/~shene/COURSES/cs3621/SLIDES/Mesh.pdf

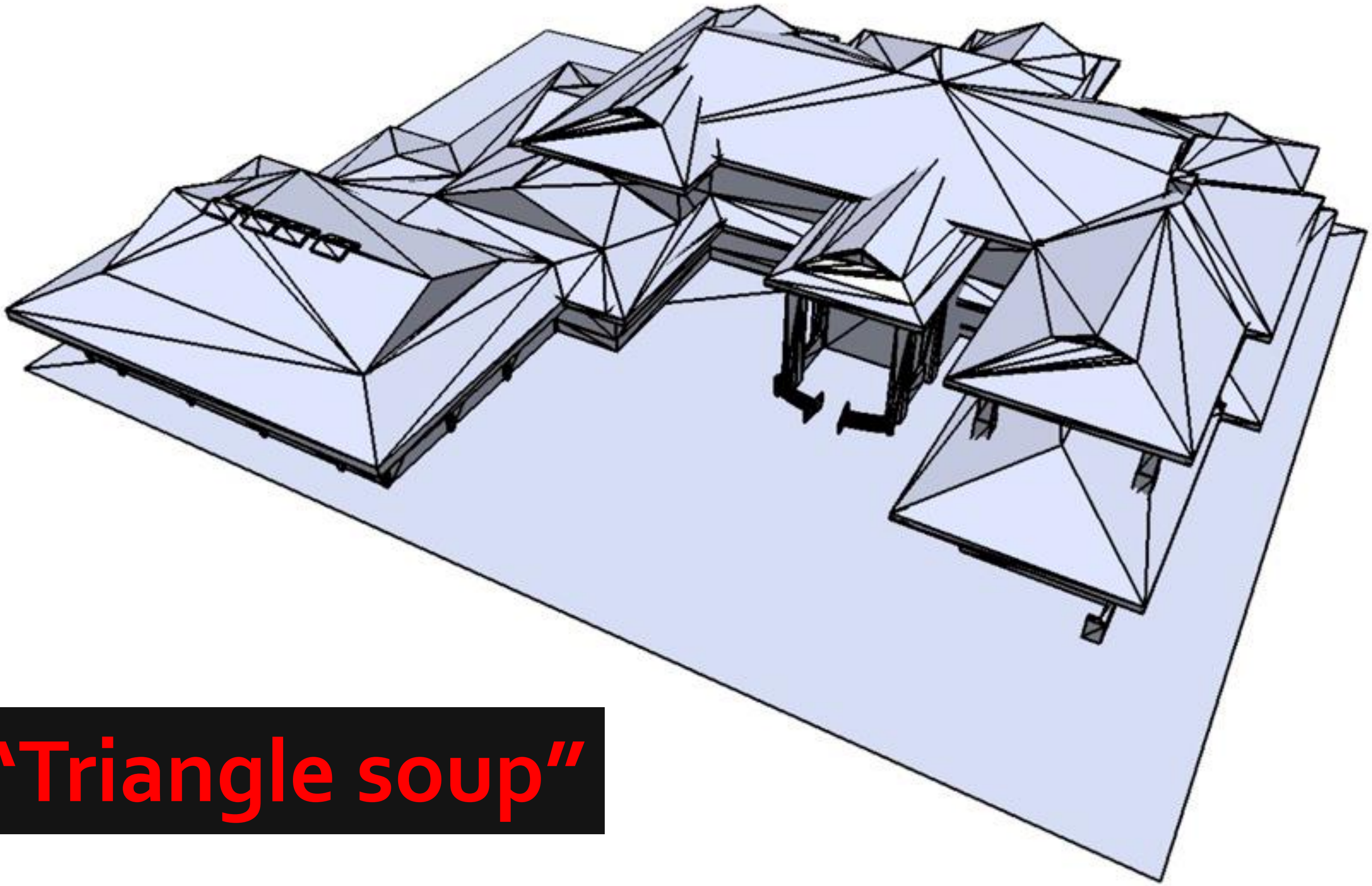
Basic Observation

Piecewise linear faces are reasonable building blocks.

Additional Advantages

- Simple to render
- Arbitrary topology possible
- Basis for subdivision, refinement

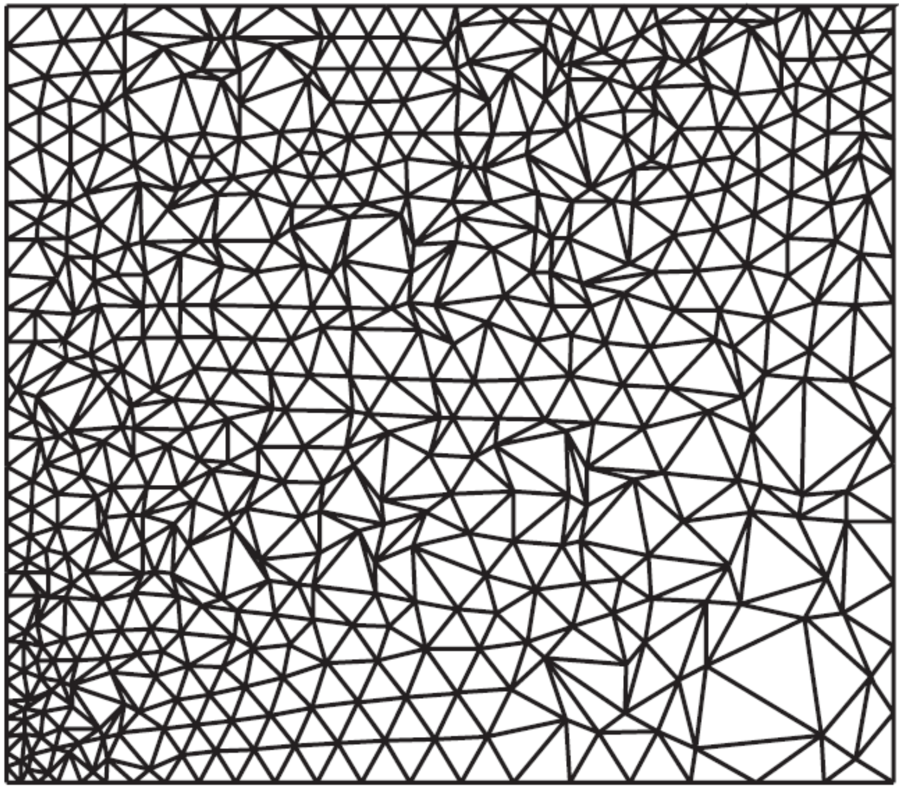
Easy-to-Violate Assumption



"Triangle soup"

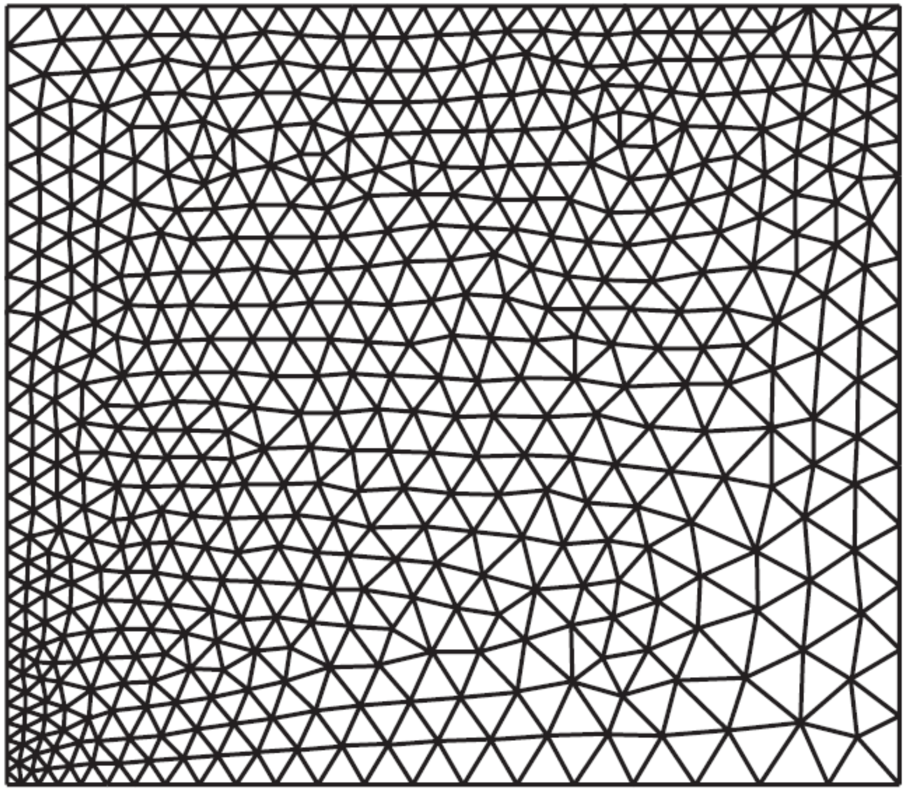
http://www.pmp-book.org/download/slides/Representations.pdf

Invalid Meshes vs. Bad Meshes



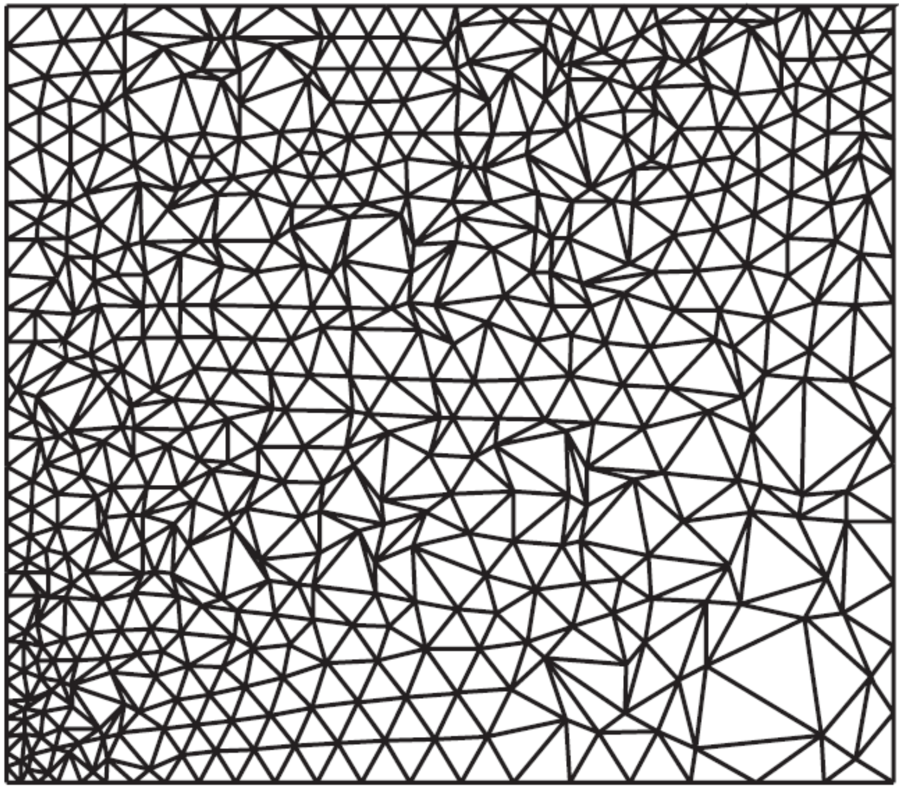
Nonuniform

areas and angles



http://www.sciencedirect.com/science/article/pii/S016887 4X06000795

Why is Meshing an Issue?



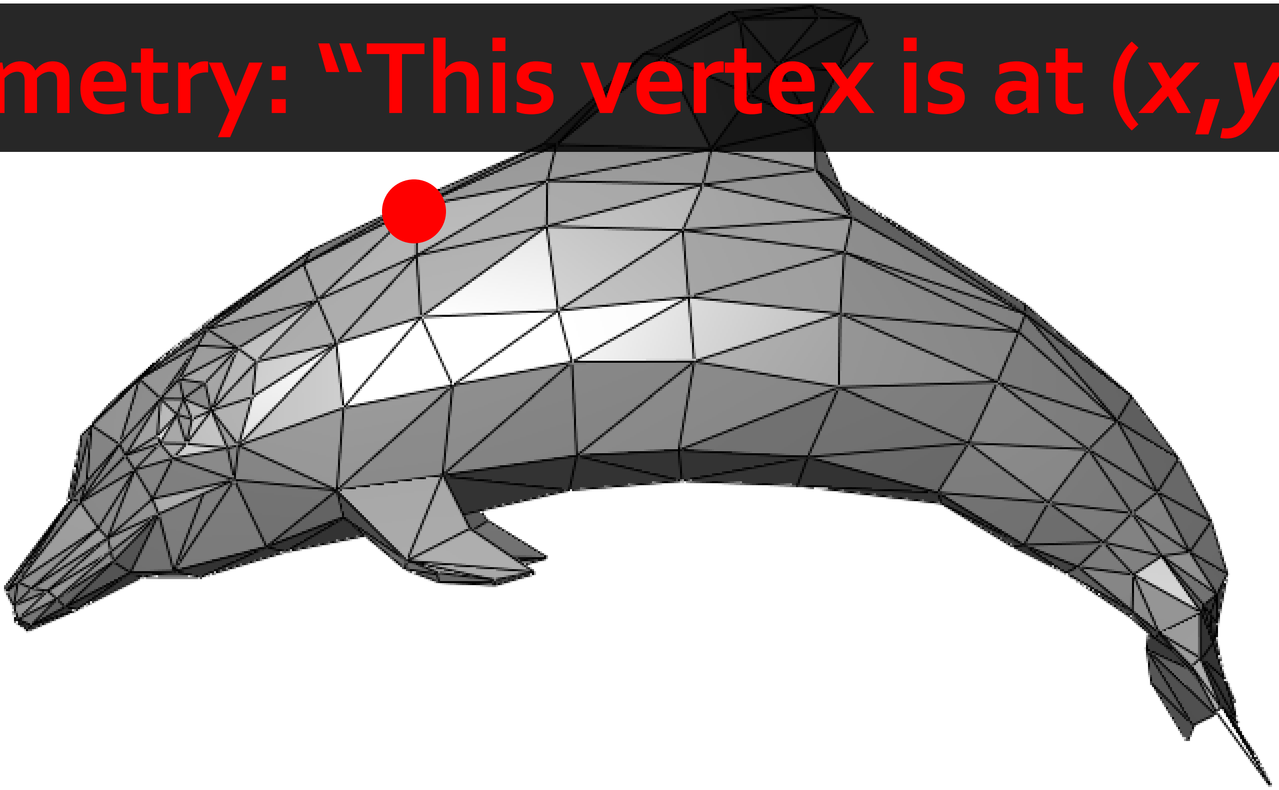
How to you interpret
one value per vertex?

http://www.sciencedirect.com/science/article/pii/S0168874X06000795

Topology [t*uh*-**pol**-*uh*-jee]: The study of geometric properties that remain invariant under certain transformations

http://dictionary.reference.com/browse/topology?s=t

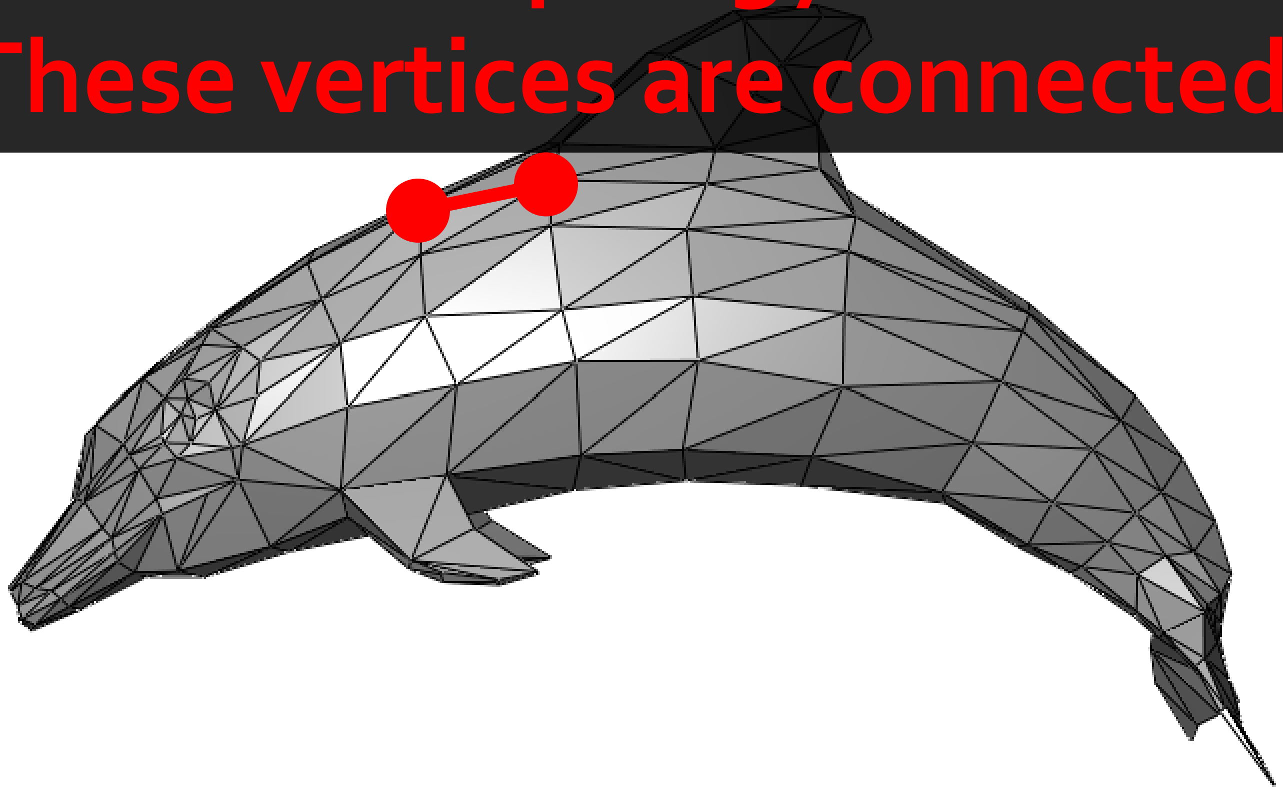
Mesh Topology vs. Geometry



http://upload.wikimedia.org/wikipedia/commons/f/fb/Dolphin_triangle_mesh.png

Mesh Topology vs. Geometry

Topology:
"These vertices are connected."



http://upload.wikimedia.org/wikipedia/commons/f/fb/Dolphin_triangle_mesh.png

Triangle Mesh

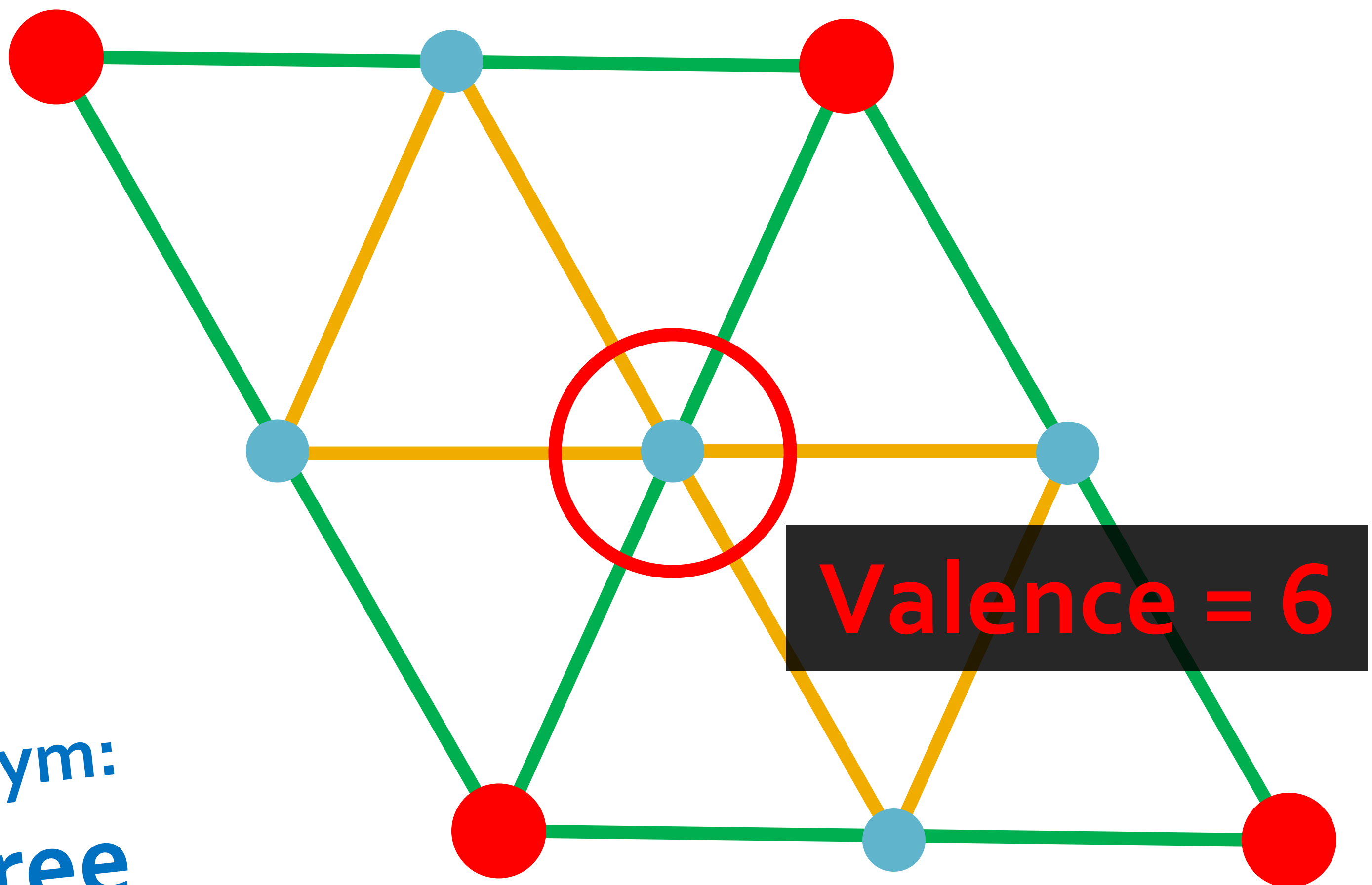
$$V = (v_1, v_2, \ldots, v_n) \subset \mathbb{R}^n$$

$$E = (e_1, e_2, \ldots, e_k) \subseteq V \times V$$

$$F = (f_1, f_2, \ldots, f_m) \subseteq V \times V \times V$$

Plus manifold conditions

Valence



Synonym:
Degree

Euler Characteristic

$$V - E + F := \chi$$

$$\chi = 2 - 2g$$

$g = 0$

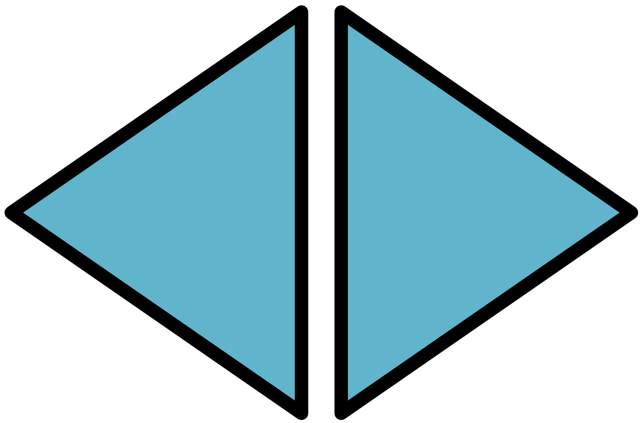
$g = 1$

$g = 2$

Consequences for Triangle Meshes

$$V - E + F := \chi$$

"Each edge is adjacent to two faces. Each face has three edges."



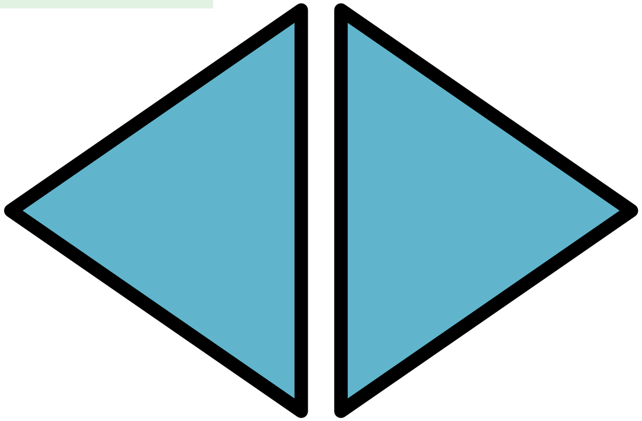
$$2E = 3F$$

Closed mesh: Easy estimates!

Consequences for Triangle Meshes

$$V - \frac{1}{2}F := \chi$$

"Each edge is adjacent to two faces. Each face has three edges."



$$2E = 3F$$

Closed mesh: Easy estimates!

Consequences for Triangle Meshes

Big number

$$V - \frac{1}{2}F := \chi$$

Small number

$$F \approx 2V$$

Closed mesh: Easy estimates!

Consequences for Triangle Meshes

$$E \approx 3V$$

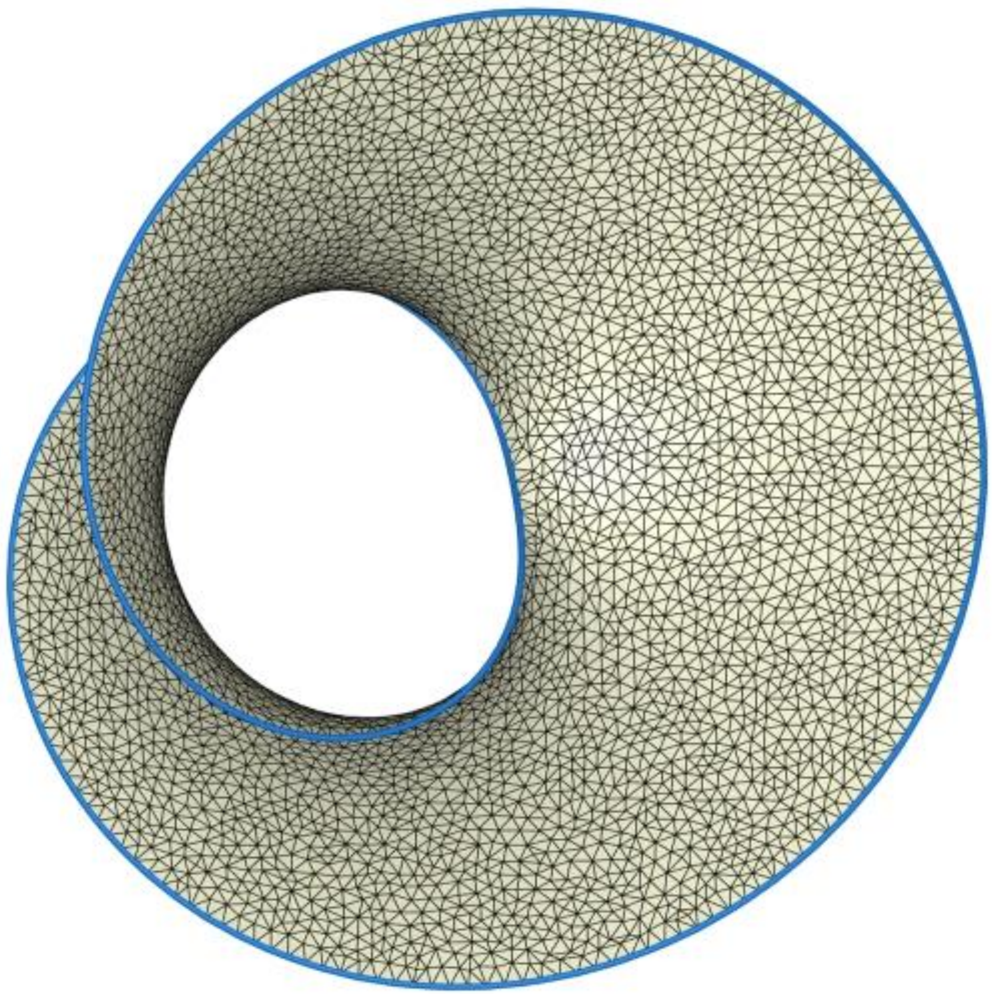
$$F \approx 2V$$

$$\text{average valence} \approx 6$$

Why?!

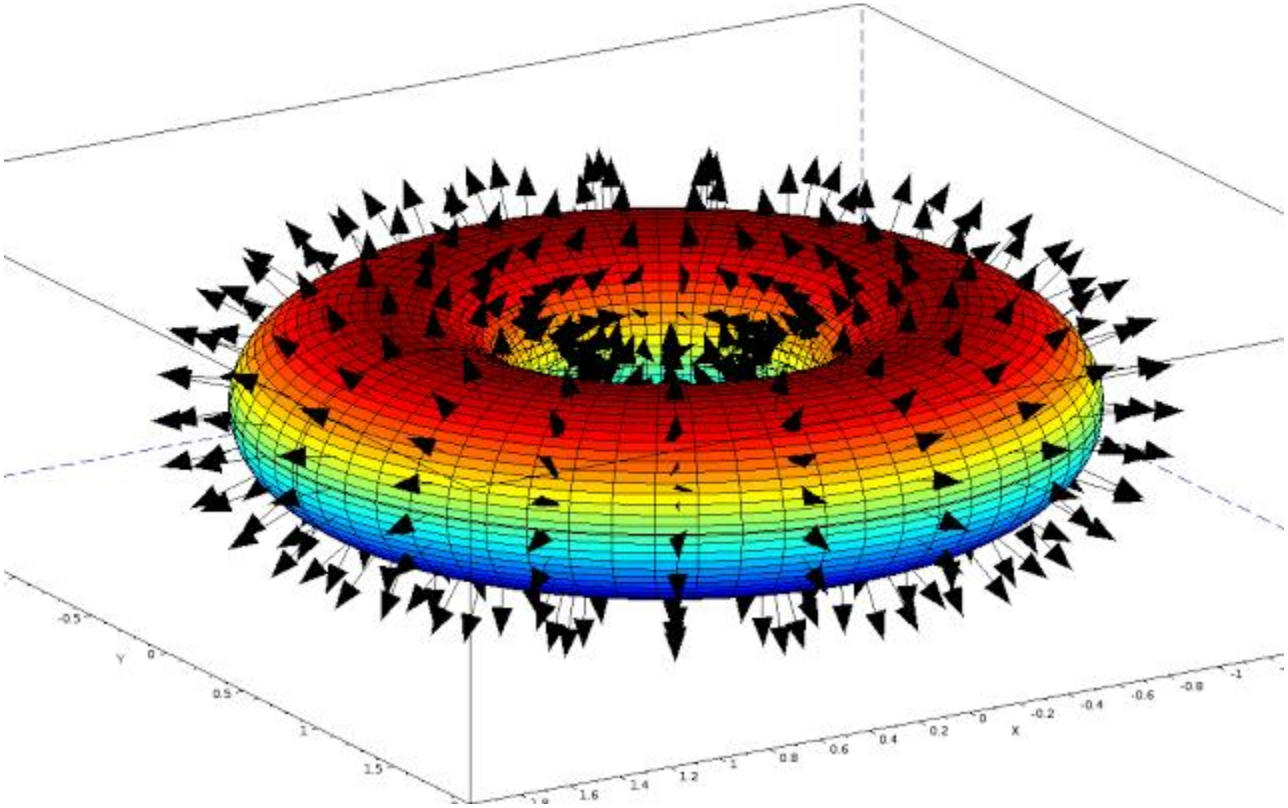
General estimates

Orientability



http://www.cse.ohio-state.edu/~tamaldey/isotopic.html

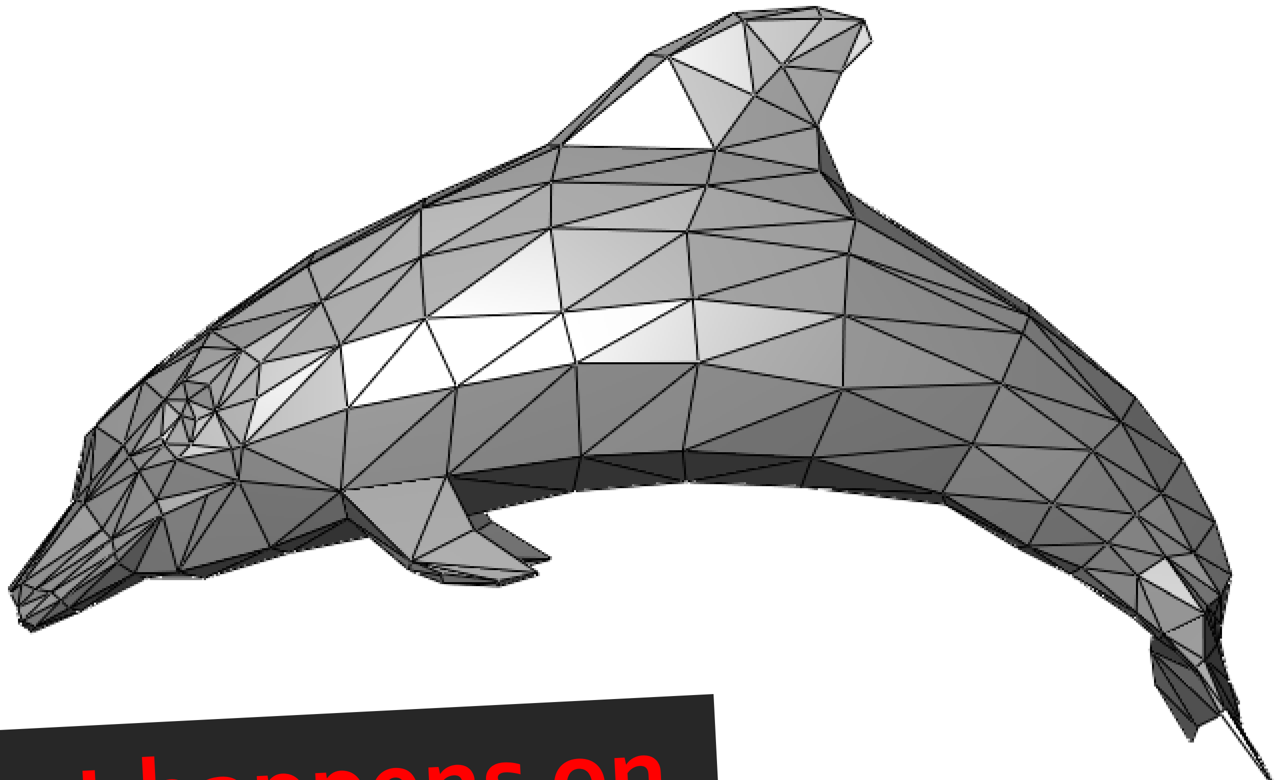
Smooth Surface Definition



https://lh3.googleusercontent.com/-njXPH7NSX5c/VV4PXu54n9I/AAAAAAAAJjM/m6TGg3ZVKGE/w640-h400-p-k/normal_tore.png

Continuous field of normal vectors

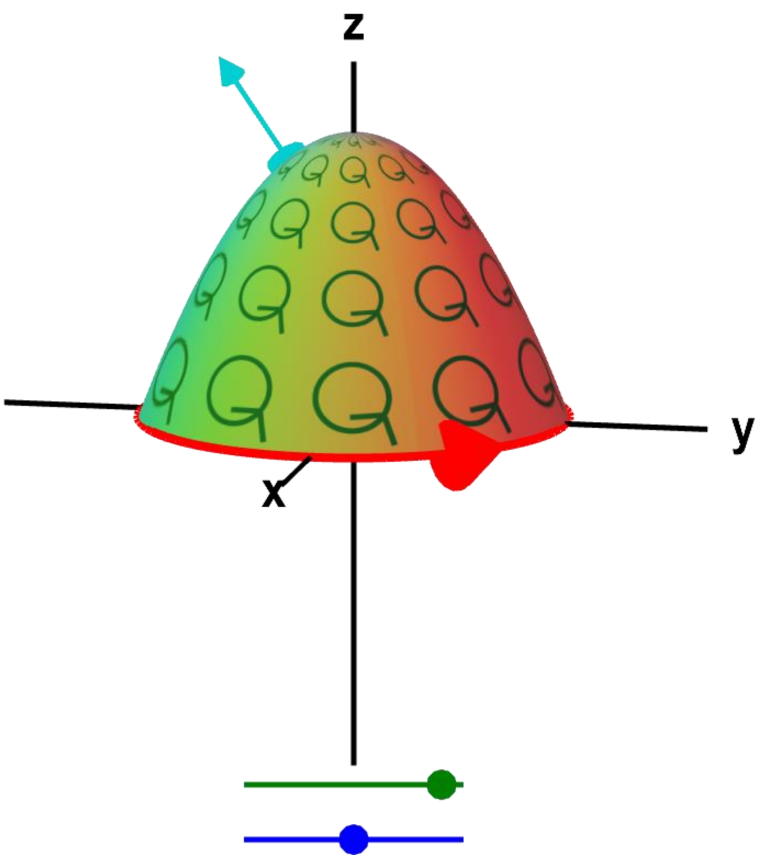
Issue on Triangle Mesh



What happens on edges/vertices?

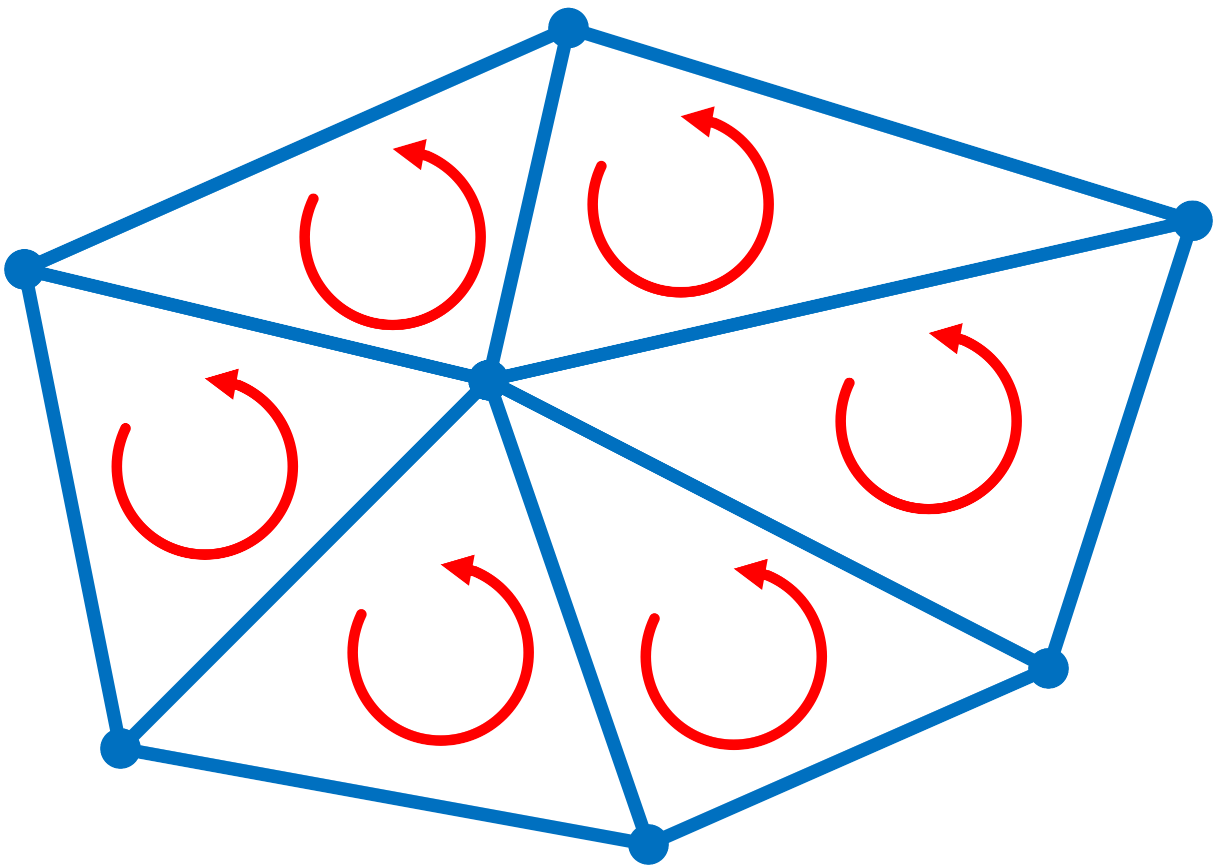
http://upload.wikimedia.org/wikipedia/commons/f/fb/Dolphin_triangle_mesh.png

Right-Hand Rule



http://viz.aset.psu.edu/gho/sem_notes/3d_fundamentals/html/3d_coordinates.html
http://mathinsight.org/stokes_theorem_orientation

Discrete Orientability



Normal field isn't continuous

Data Structures for Surfaces

Must represent geometry <u>and</u> topology.

Simplest Format

```
x1 y1 z1 / x2 y2 z2 / x3 y3 z3
x1 y1 z1 / x2 y2 z2 / x3 y3 z3
x1 y1 z1 / x2 y2 z2 / x3 y3 z3
x1 y1 z1 / x2 y2 z2 / x3 y3 z3
x1 y1 z1 / x2 y2 z2 / x3 y3 z3
```

No topology!

```
glBegin(GL_TRIANGLES)
```

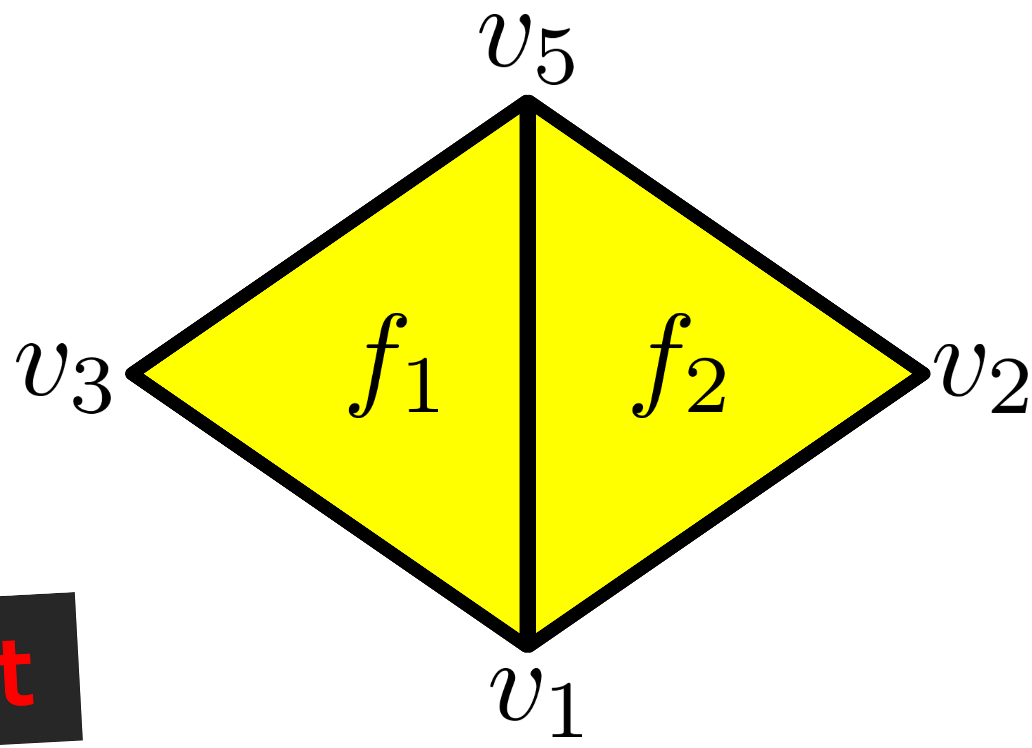
CS 468 2011 (M. Ben-Chen), other slides

Triangle soup

Factor Out Vertices

```
f 1 5 3
f 5 1 2
…
v 0.2 1.5 3.2
v 5.2 4.1 8.9
…
```

.obj format



CS 468 2011 (M. Ben-Chen), other slides

Shared vertex structure

Simple Mesh Smoothing

```
for i=1 to n
  for each vertex v
    v = .5*v +
      .5*(average of neighbors);
```

Typical Queries

- Neighboring vertices to a vertex
- Neighboring faces to an edge
- Edges adjacent to a face
- Edges adjacent to a vertex
- …

Mostly localized

Typical Queries

- Neighboring vertices to a vertex
- Neighboring faces to an edge
- Edges adjacent to a face
- Edges adjacent to a vertex
- …

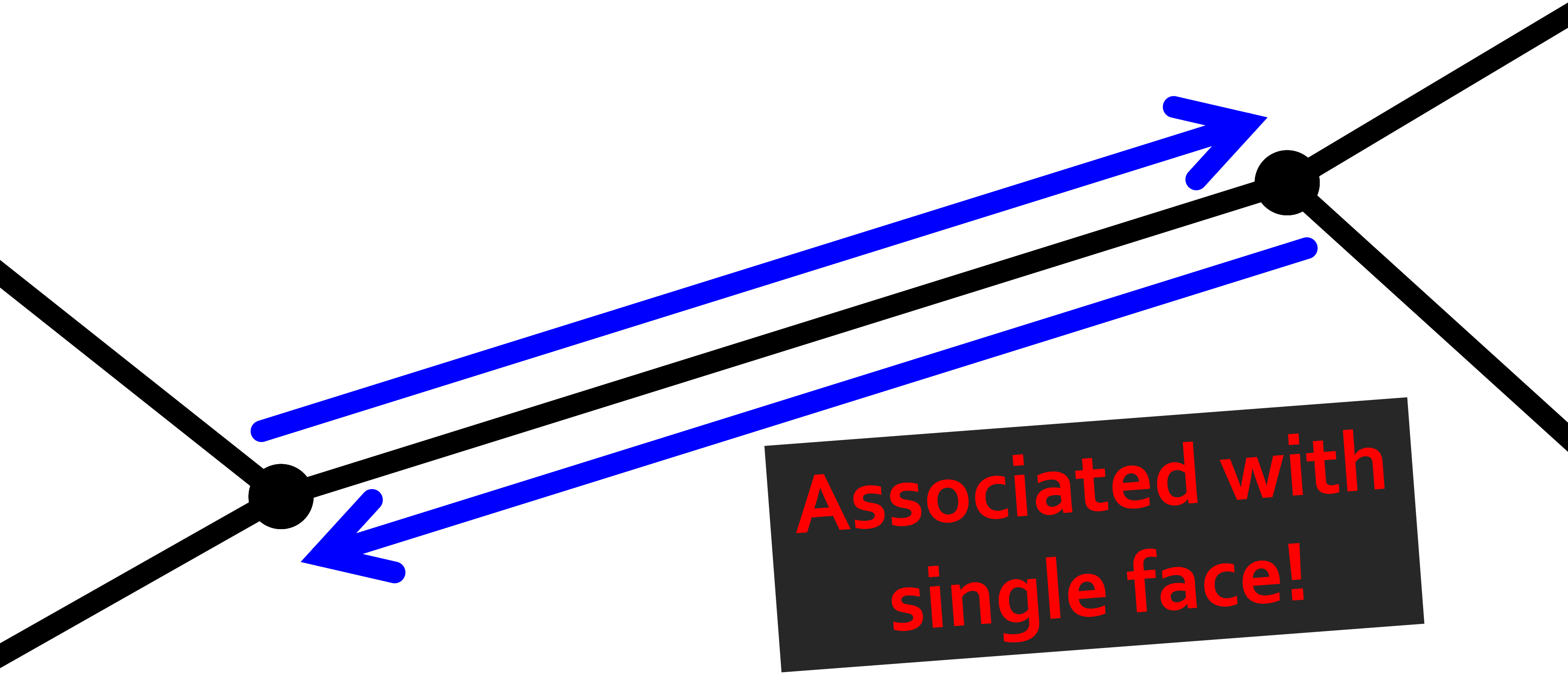
Mostly localized

Pieces of Halfedge Data Structure

- Vertices
- Faces
- Half-edges

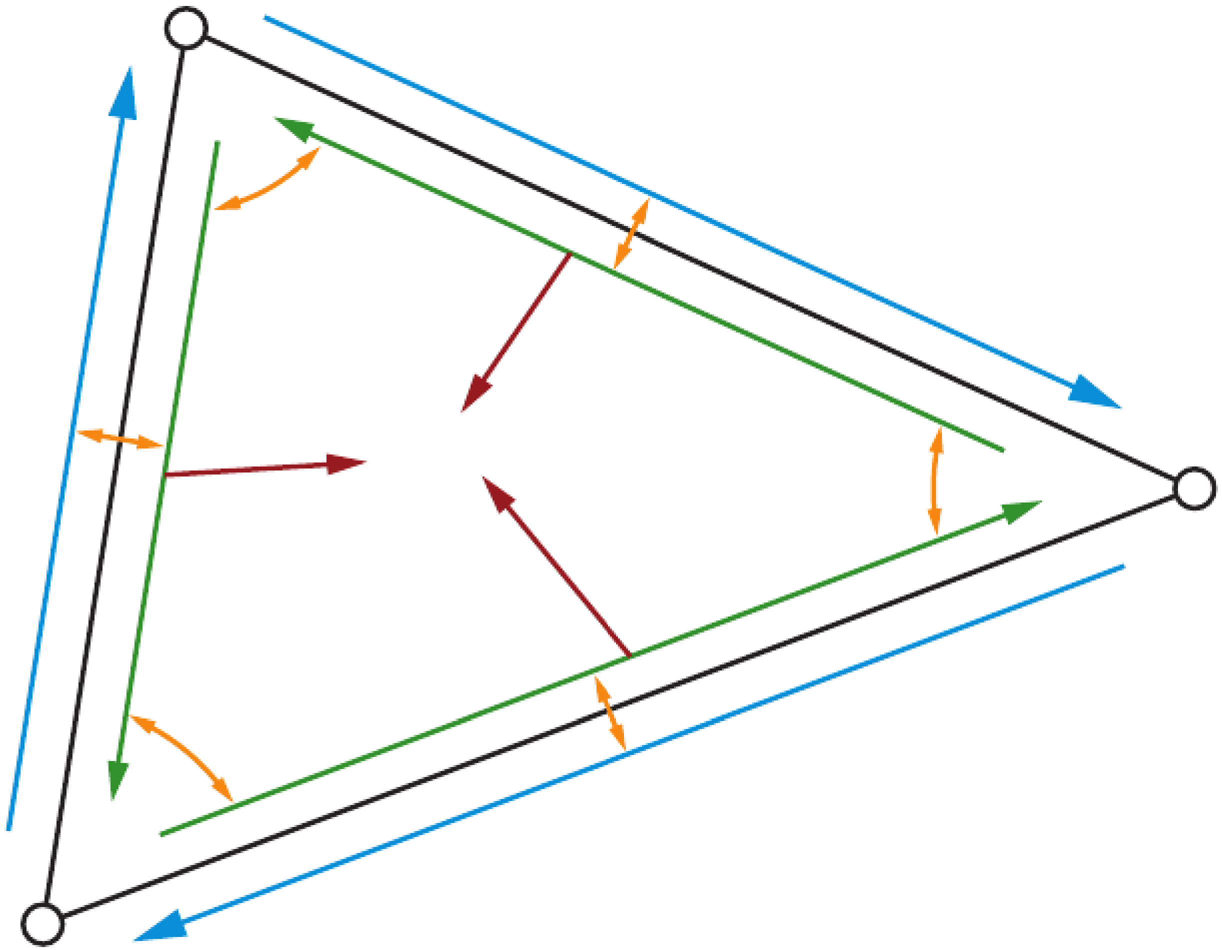
Structure tuned for meshes

Halfedge?



Oriented edge

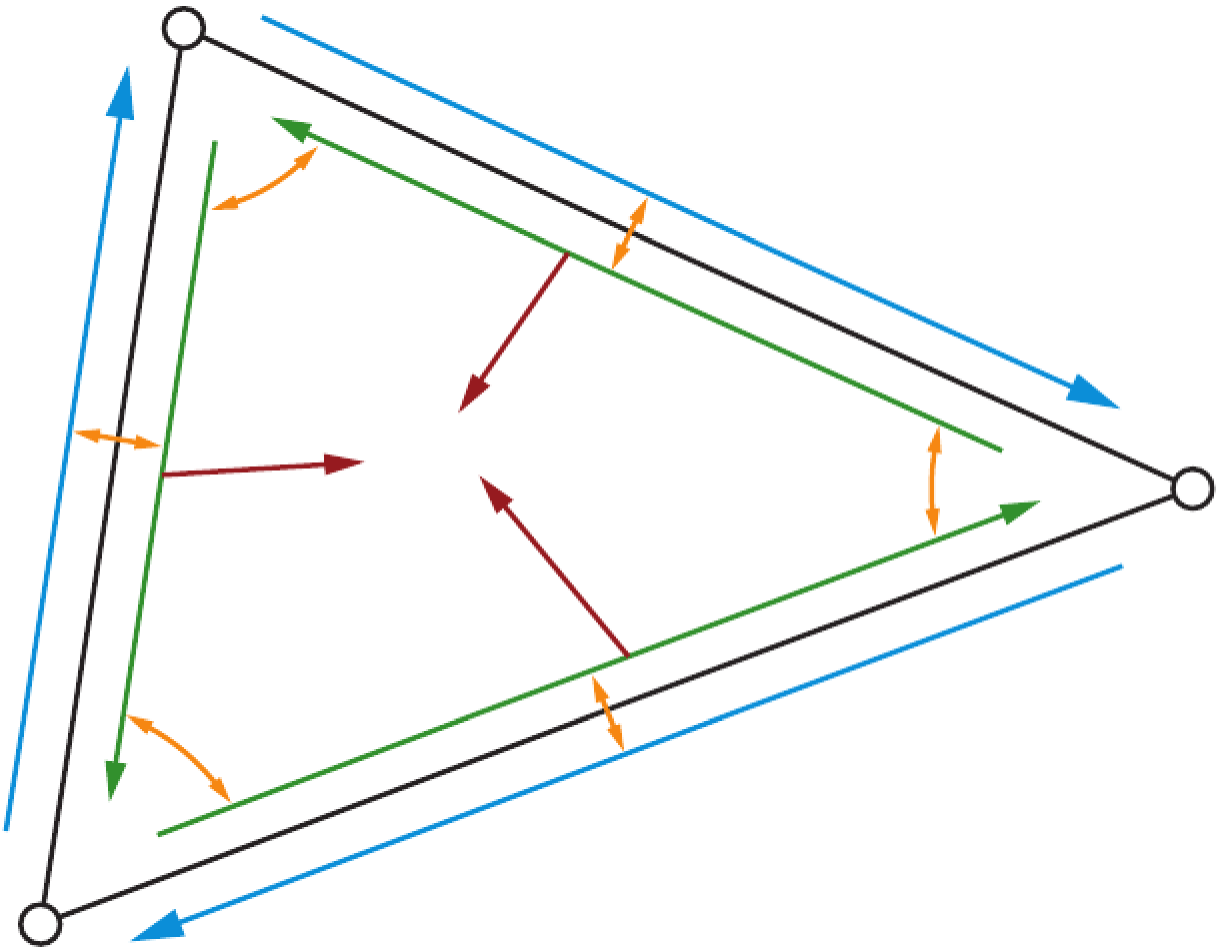
Halfedge Data Types



Vertex stores:

- Arbitrary outgoing halfedge

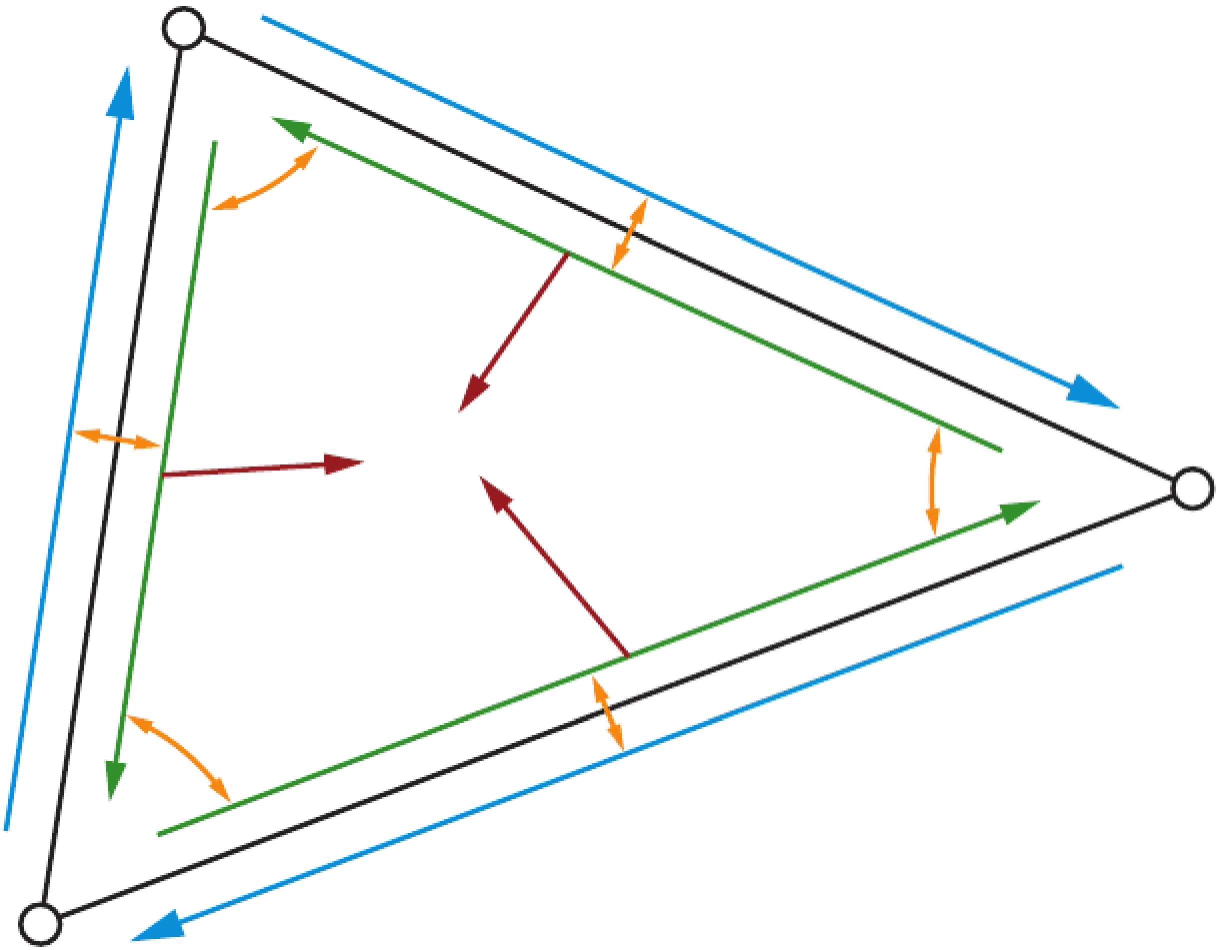
Halfedge Data Types



Face stores:

- Arbitrary adjacent halfedge

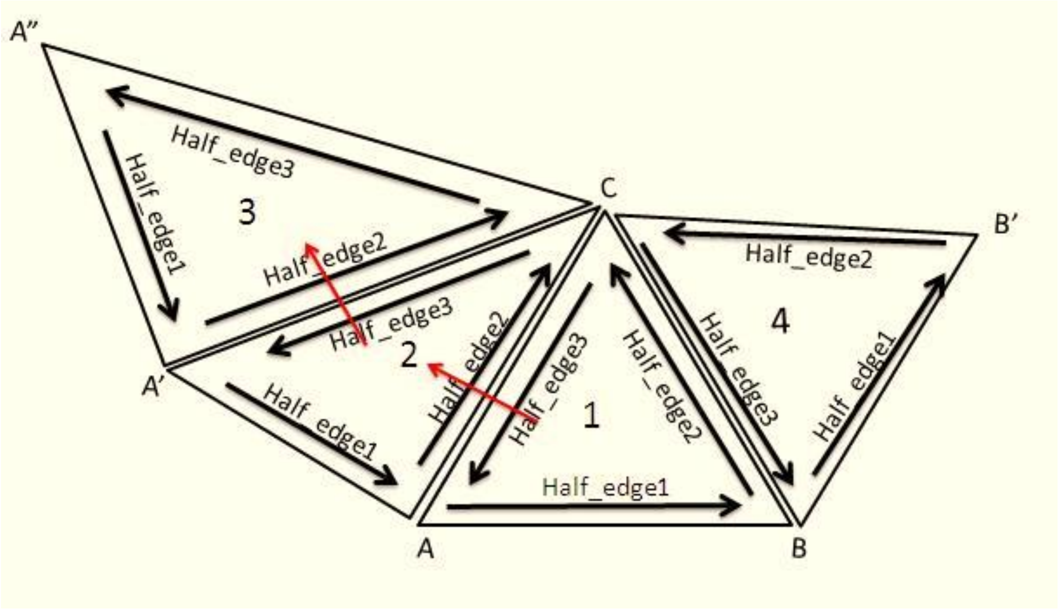
Halfedge Data Types



Halfedge stores:

- Flip
- Next
- Face
- Vertex

Iterating Over Vertex Neighbors




```
Iterate(v):
startEdge = v.out;
e = startEdge;
do
    process(e.flip.from)
    e = e.flip.next
while e != startEdge
```

https://sympa.inria.fr/sympa/arc/cgal-discuss/2008-05/msg00054.html

Only Scratching the Surface

Eurographics Symposium on Geometry Processing (2005)
M. Desbrun, H. Pottmann (Editors)

Streaming Compression of Triangle Meshes

Martin Isenburg[1†] Peter Lindstrom[2] Jack Snoeyink[1]

[1] University of North Carolina at Chapel Hill [2] Lawrence Livermore National Labs

EUROGRAPHICS 2011 / M. Chen and O. Deussen
(Guest Editors)

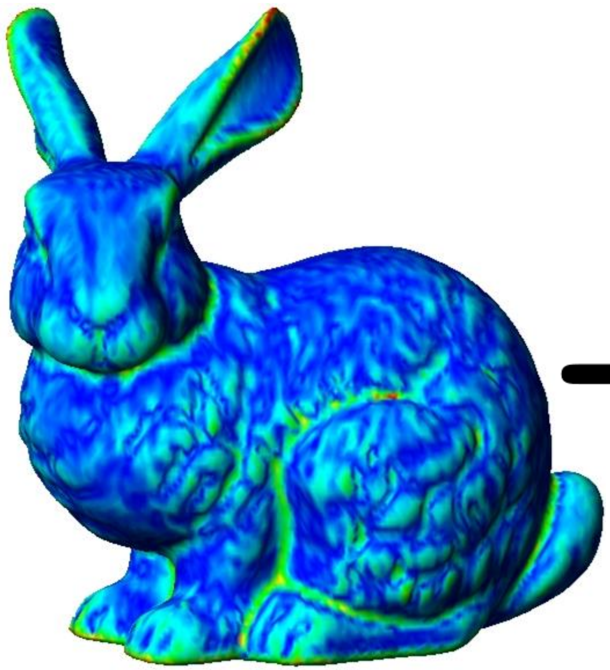
Volume 30 (2011), Number 2

SQuad: Compact Representation for Triangle Meshes

Topraj Gurung[1], Daniel Laney[2], Peter Lindstrom[2], Jarek Rossignac[1]

[1]Georgia Institute of Technology
[2]Lawrence Livermore National Laboratory

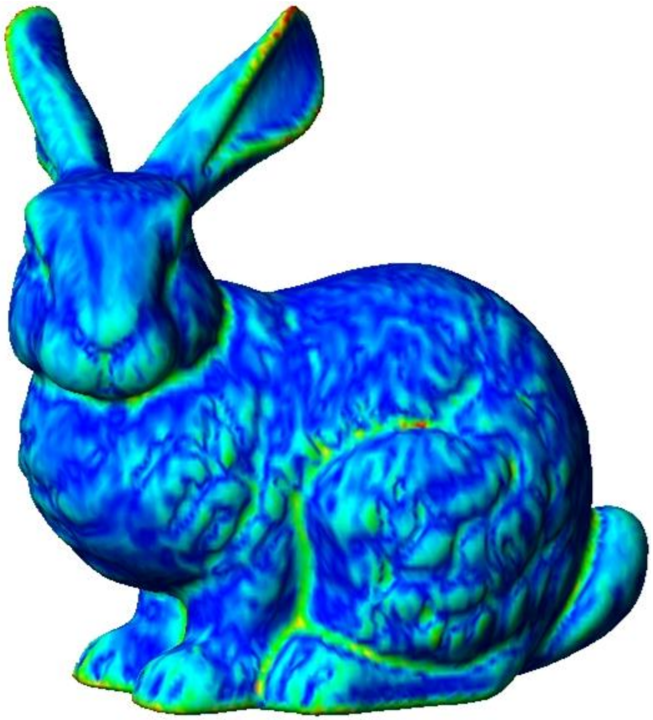
Scalar Functions

$f :$  $\longrightarrow \mathbb{R}$

http://www.ieeta.pt/polymeco/Screenshots/PolyMeCo_OneView.jpg

Map points to real numbers

Discrete Scalar Functions



$$f \in \mathbb{R}^{|V|}$$

http://www.ieeta.pt/polymeco/Screenshots/PolyMeCo_OneView.jpg

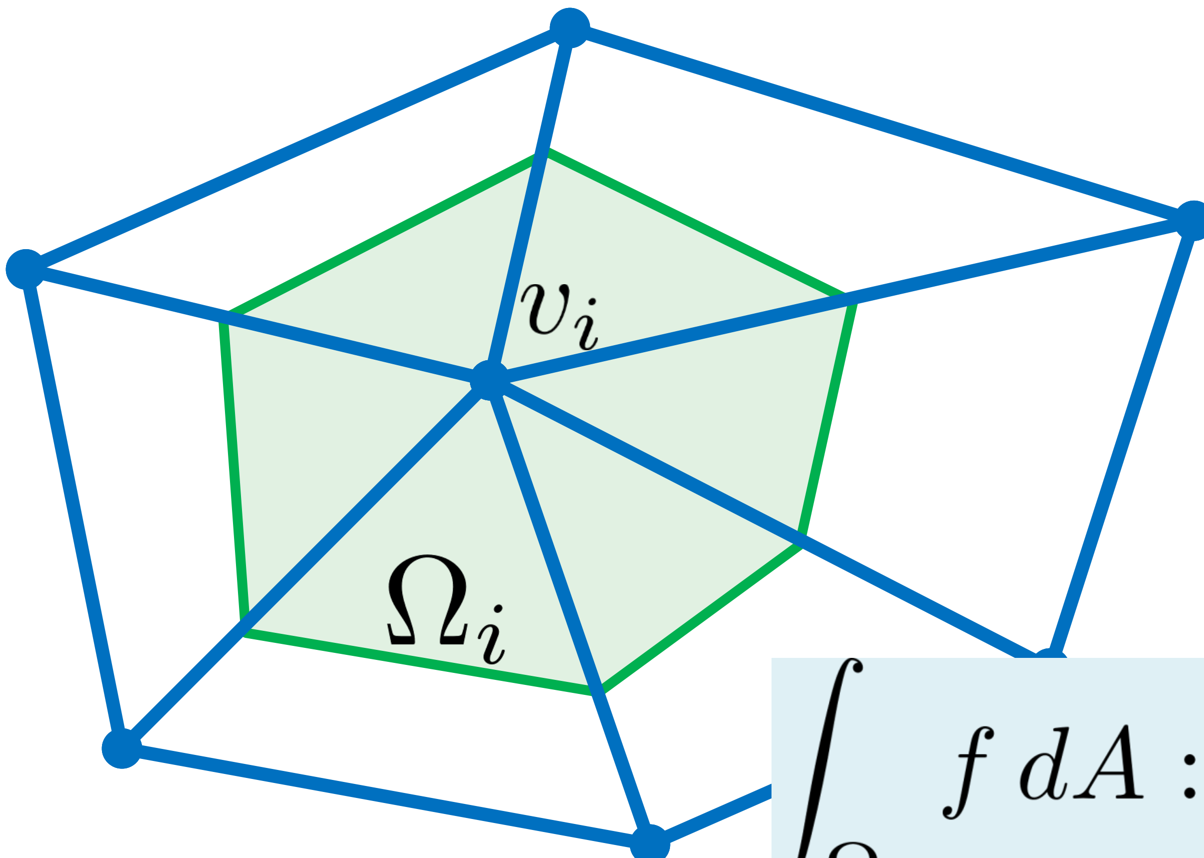
Map vertices to real numbers

Question

What is the integral of *f*?

$$\int_M f \, dA$$

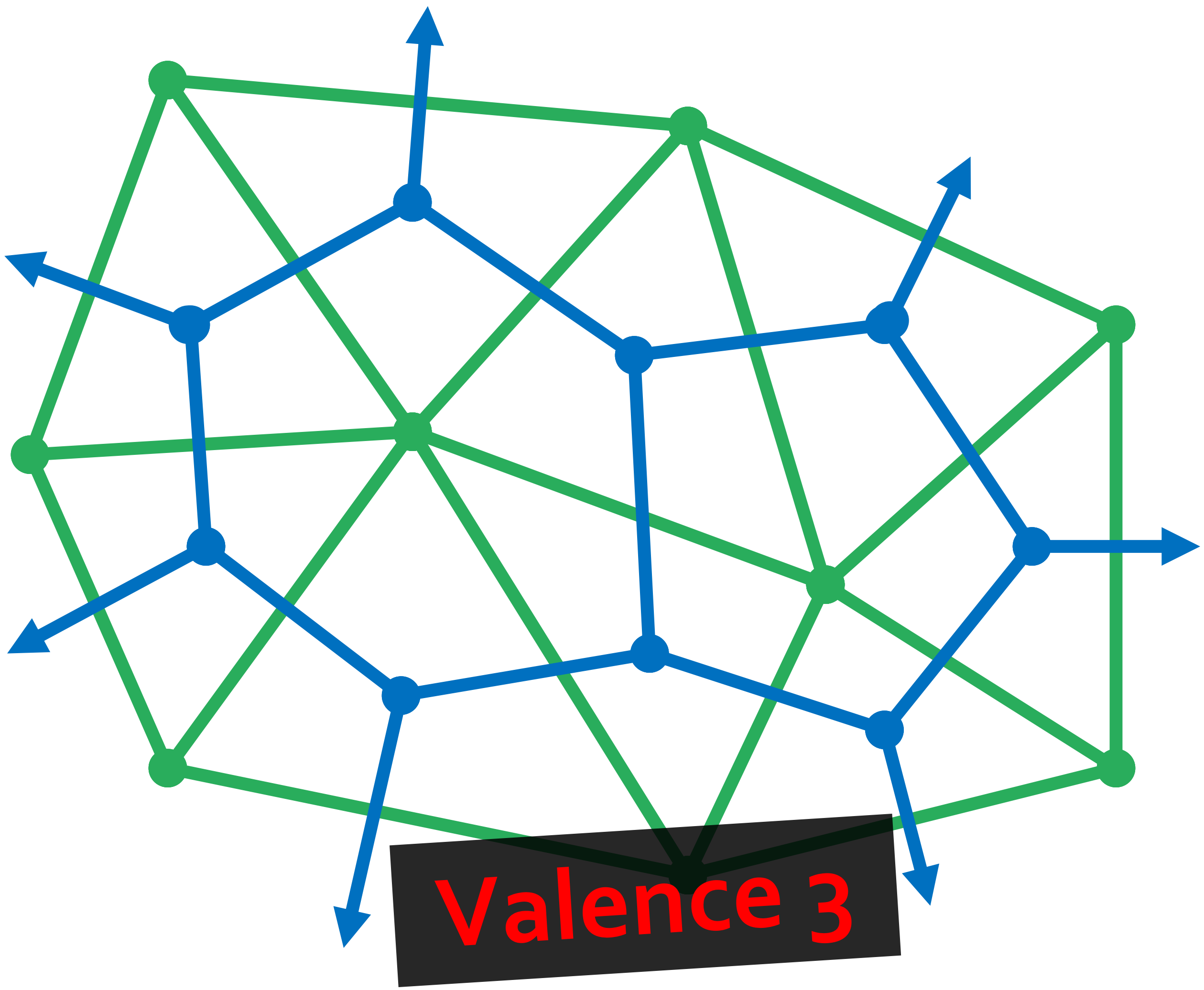
Dual Cell



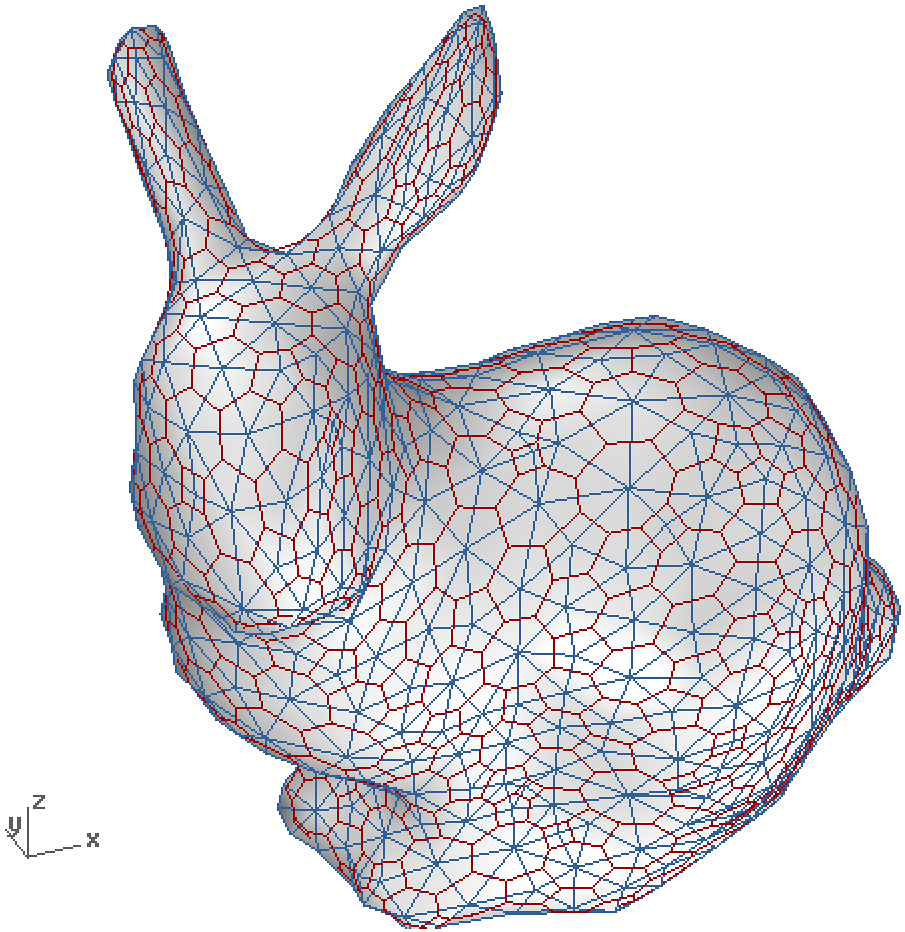
$$\int_{\Omega_i} f \, dA := f_i |\Omega_i|$$

Discrete version of dA

Dual Complex

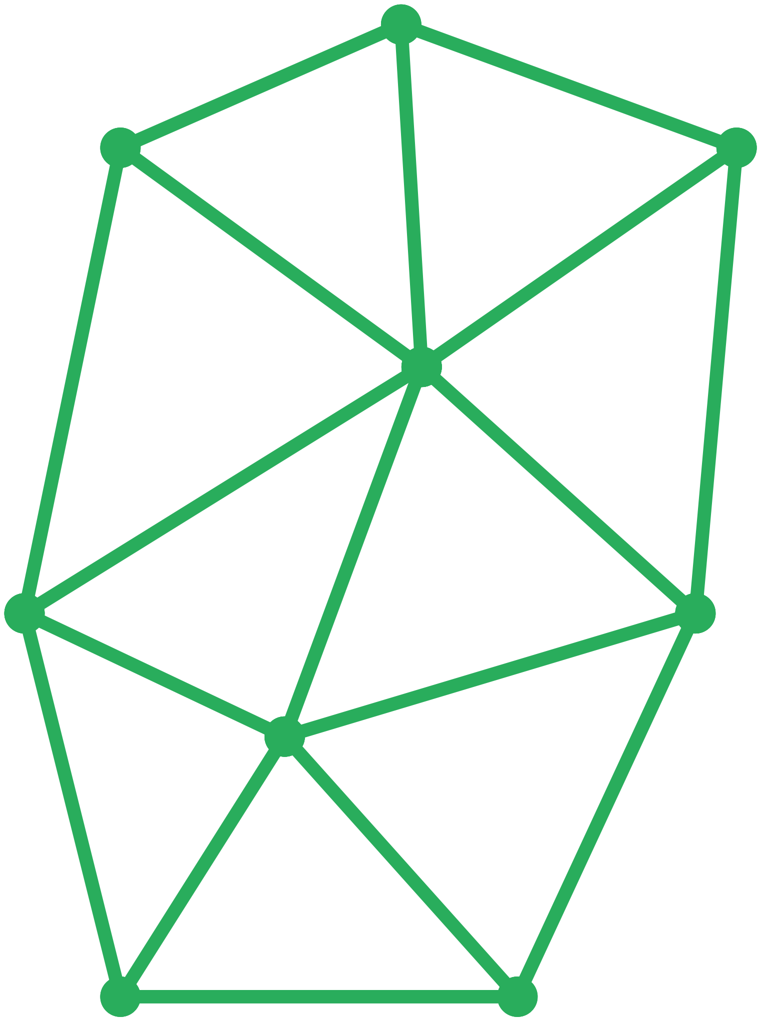


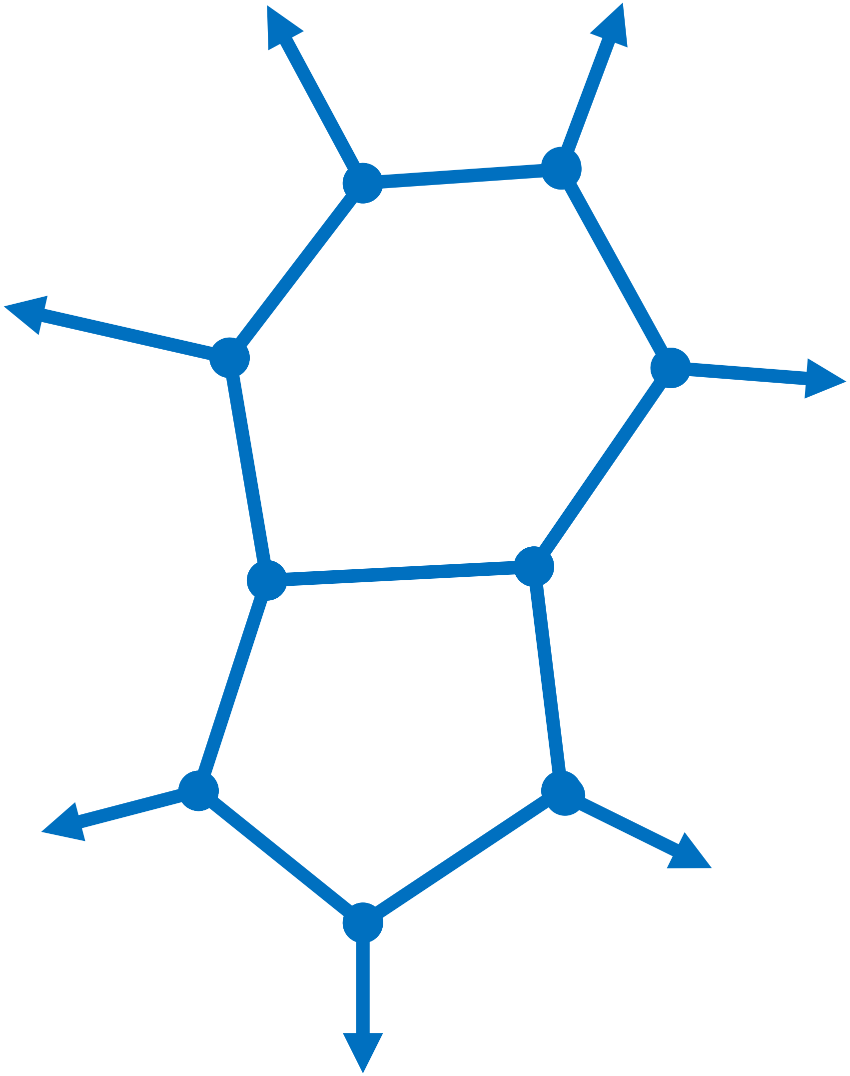
One Surface, Two Meshes



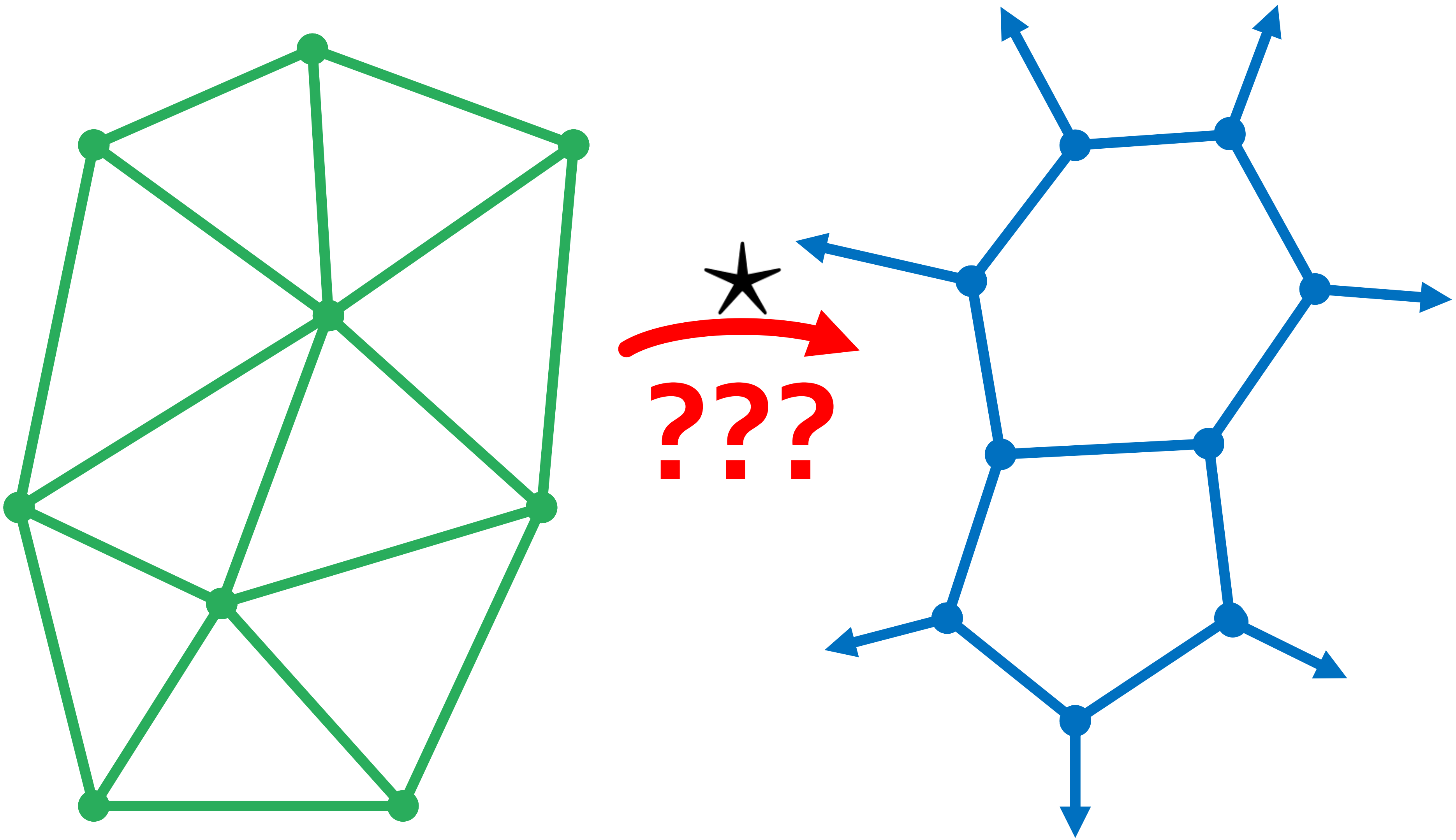
http://www.grasshopper3d.com/group/kangaroo/forum/topics/isosurface-dynamic-remeshing

One Surface, Two Halfedges

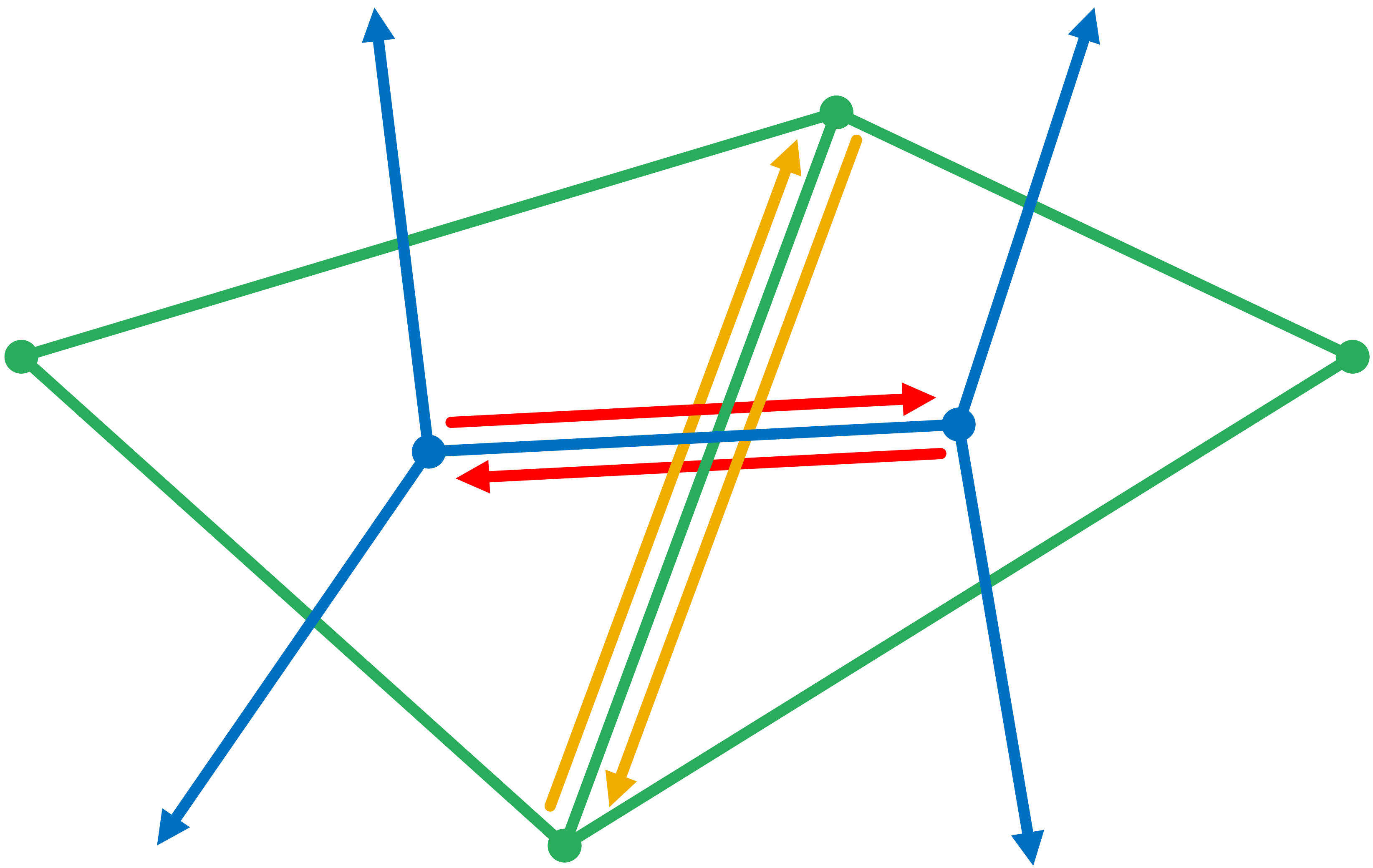




Missing Operation

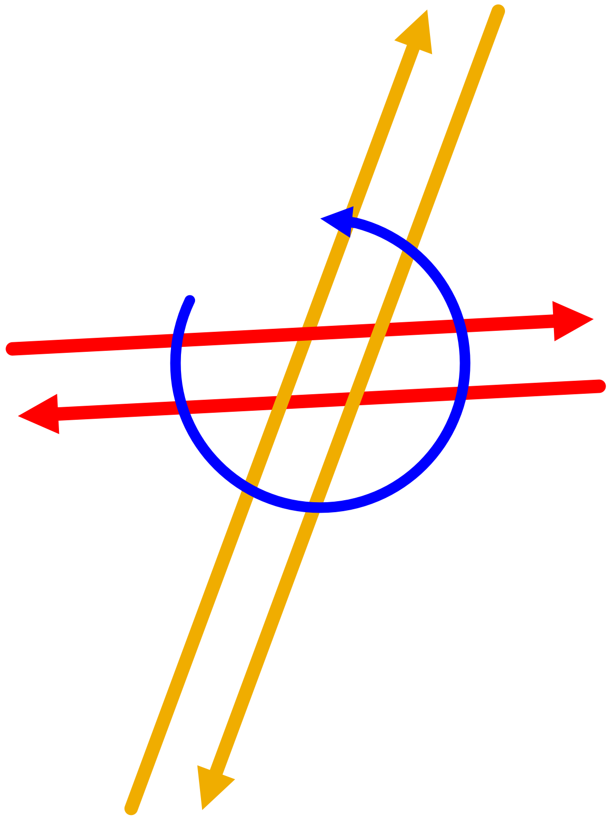


Quad Edge

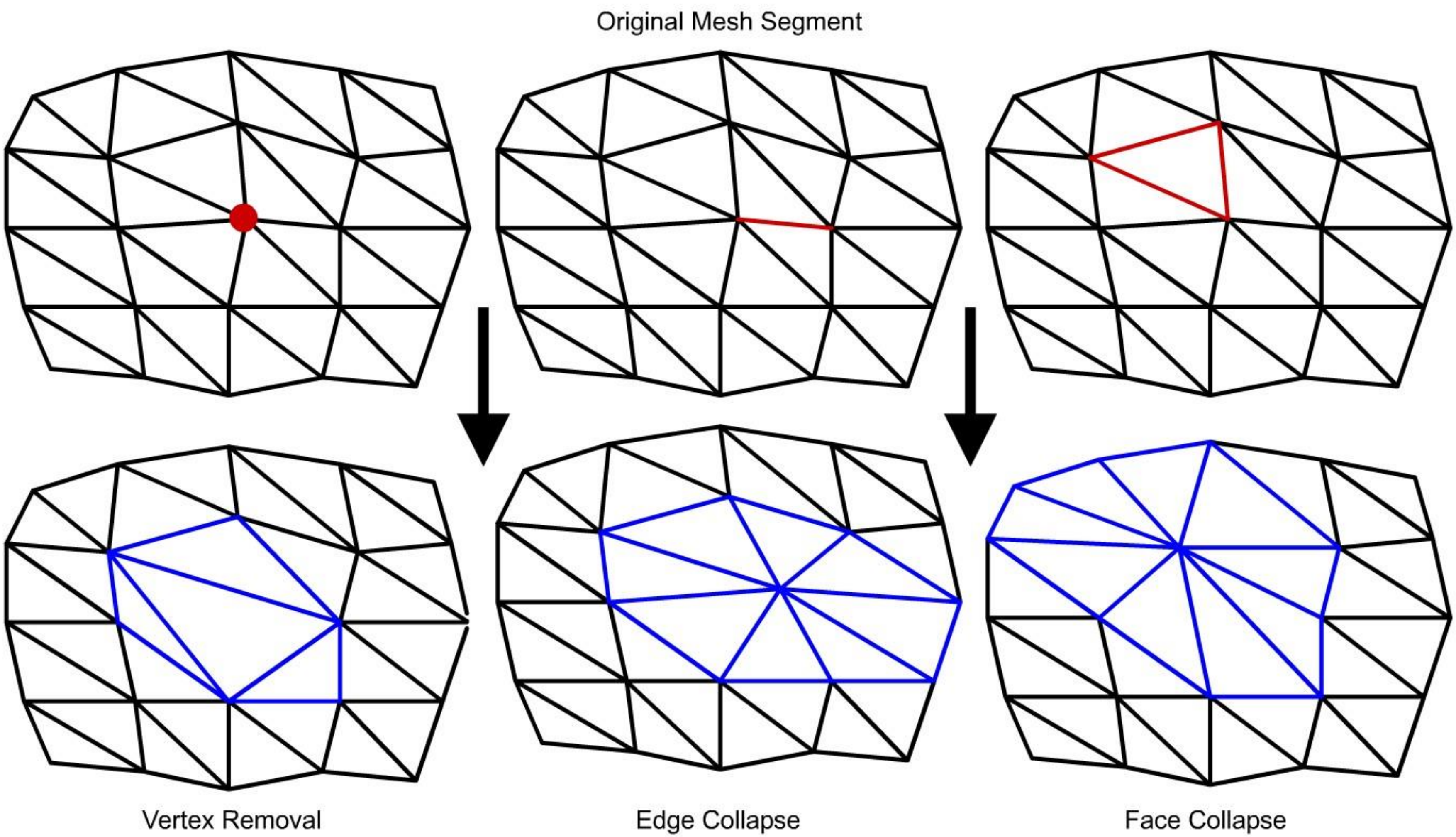


Rotation Operation

$$e \to \mathrm{Rot} \to \mathrm{Rot} = e \to \mathrm{Flip}$$

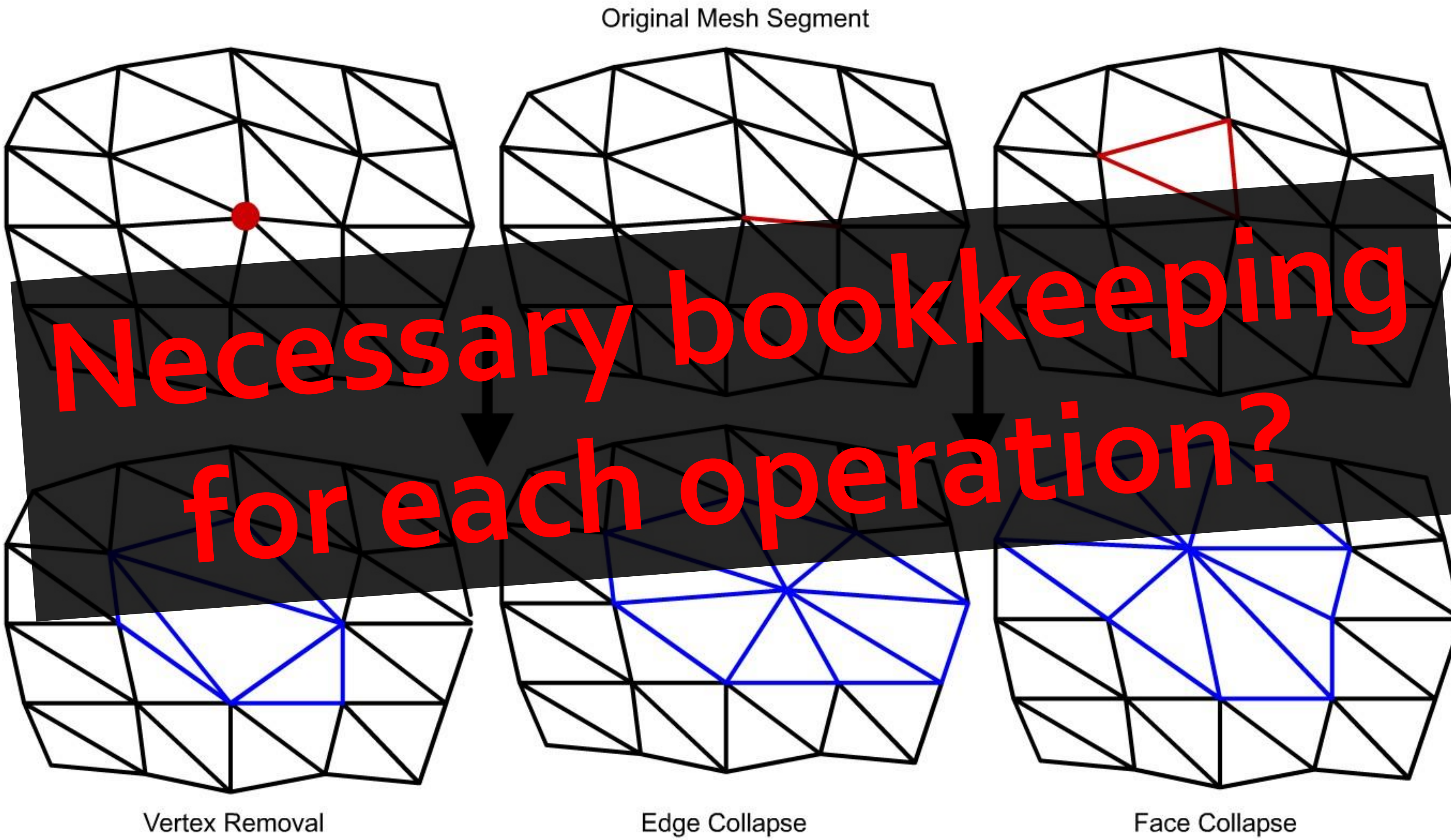


Topological Operations



http://johnrichie.com/V2/richie/simp/images/types1.jpg

Topological Operations

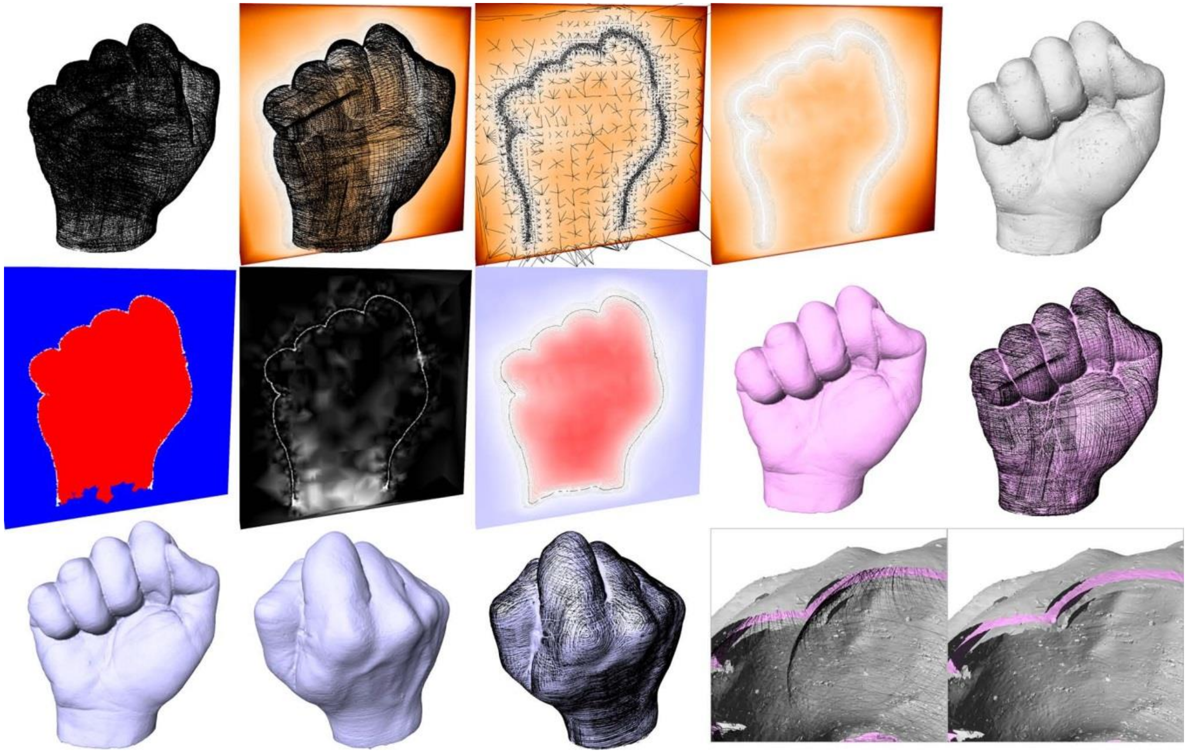


http://johnrichie.com/V2/richie/simp/images/types1.jpg

Take-Away

Complex data structures enable simpler traversal at cost of more bookkeeping.

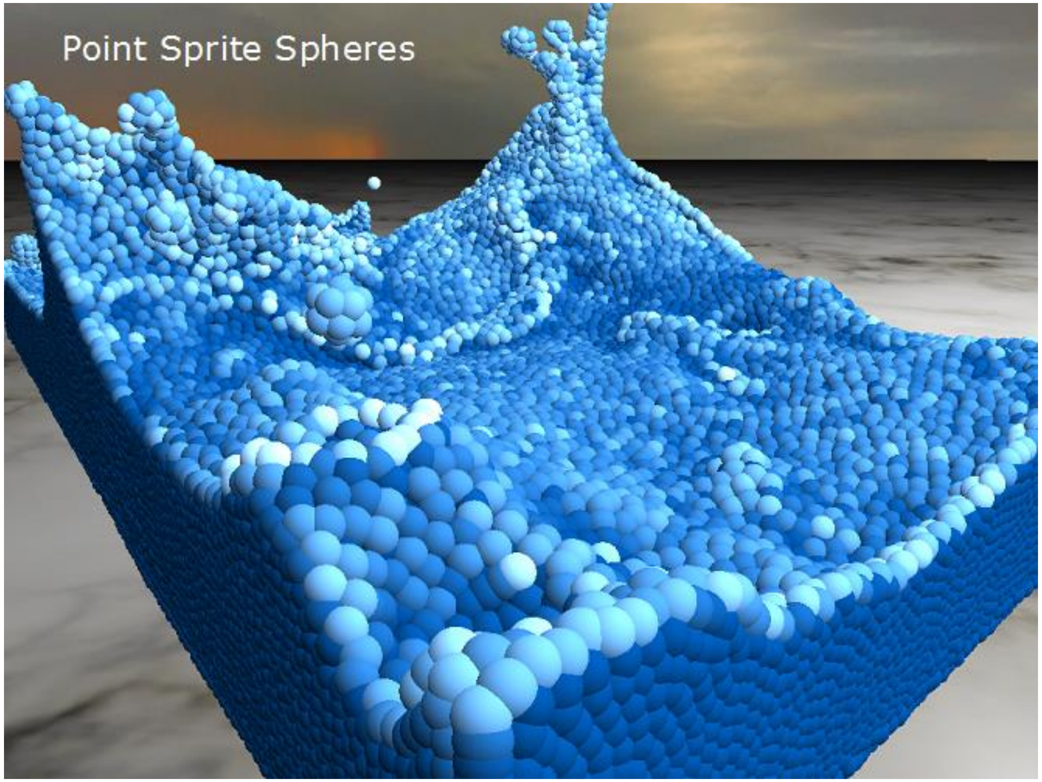
Not the Only Model



http://www.cs.umd.edu/class/spring2005/cmsc828v/papers/mpu_implicits.pdf ftp://ftp-sop.inria.fr/geometrica/alliez/signing.pdf

Implicit surfaces

Not the Only Model



http://www.itsartmag.com/features/cgfluids/
https://developer.nvidia.com/content/fluid-simulation-alice-madness-returns

Smoothed-particle hydrodynamics

Not the Only Model

AN IMPLICIT SURFACE TENSION MODEL

J. I. Hochstein* and T. L. Williams**
The University of Memphis
Memphis, Tennessee

ABSTRACT

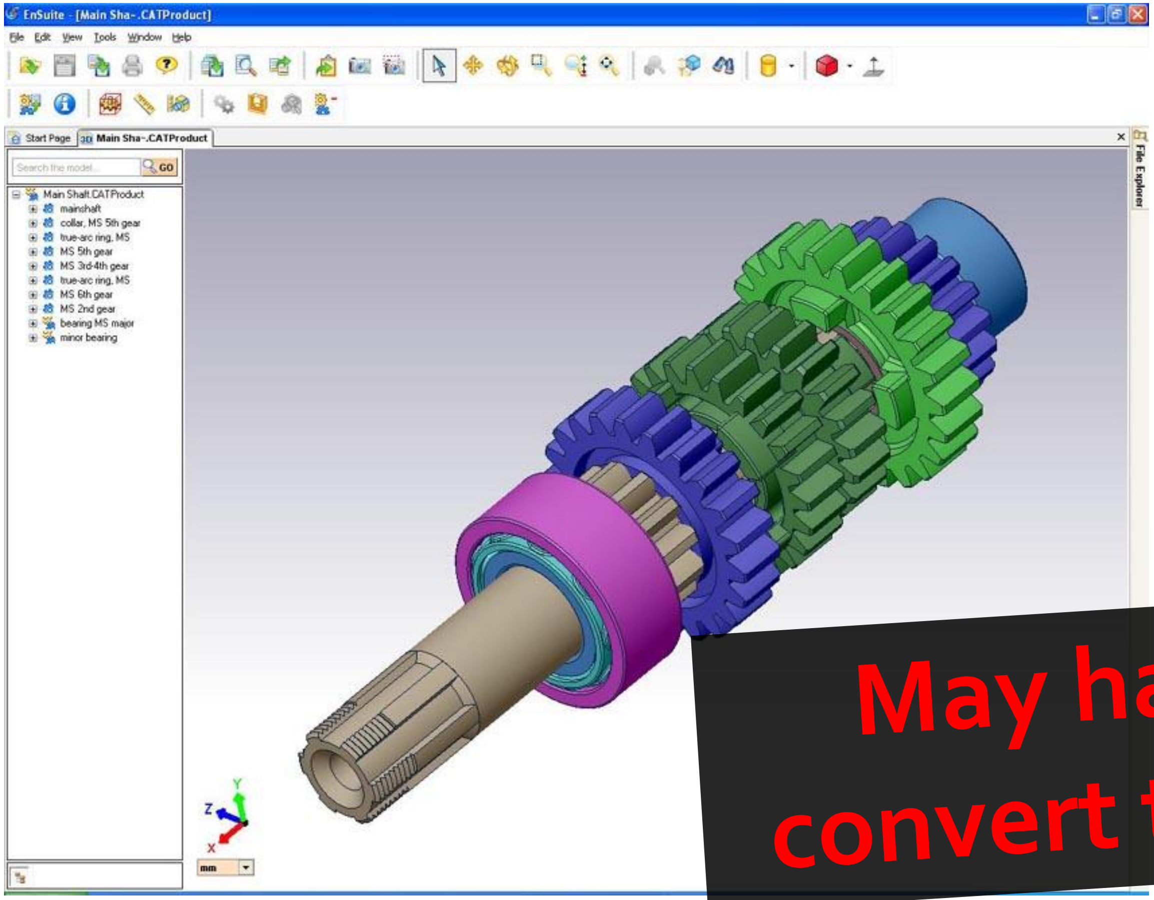
A new implicit model for surface tension at a two-fluid interface is proposed for use in computational models of flows with free surfaces and its performance is compared to an existing explicit model. The new model is based on an evolution equation for surface curvature that includes the influence of advection as well as surface tension. A detailed development of the new model is presented as are the details of the computational implementation. The performance of the new model is compared to an existing explicit model by using both models to predict the surface dynamics of several two-dimensional configurations. It is concluded that the new implicit surface tension model does perform better for configurations with a large surface tension coefficient. It is shown that, for several cases, the time step size is no longer limited by surface tension stability considerations (as it was using the explicit model), but rather by other limitations inherent in the existing volume advection algorithm.

INTRODUCTION

Incompressible flows with a free-surface exist in many industrial applications. Some examples include fuel atomization in internal combustion engines, droplet size control in ink-jet printers, formation of lead shot, control of liquid spacecraft propellant in low gravity, and the spinning of synthetic fibers. The technology for some of these applications has been developed by heavily relying on experimental study of the specific process involved. For others, such as spacecraft propellant management, experimental studies are prohibitively expensive and the ability to computationally model these process is essential for their development.

The modeling of flows with a free surface presents challenges unlike other types of flow problems in that a boundary condition must be applied at the free surface which is often in a transient state and irregularly shaped. This problem is exacerbated when the force due to surface tension

Obtaining Geometry

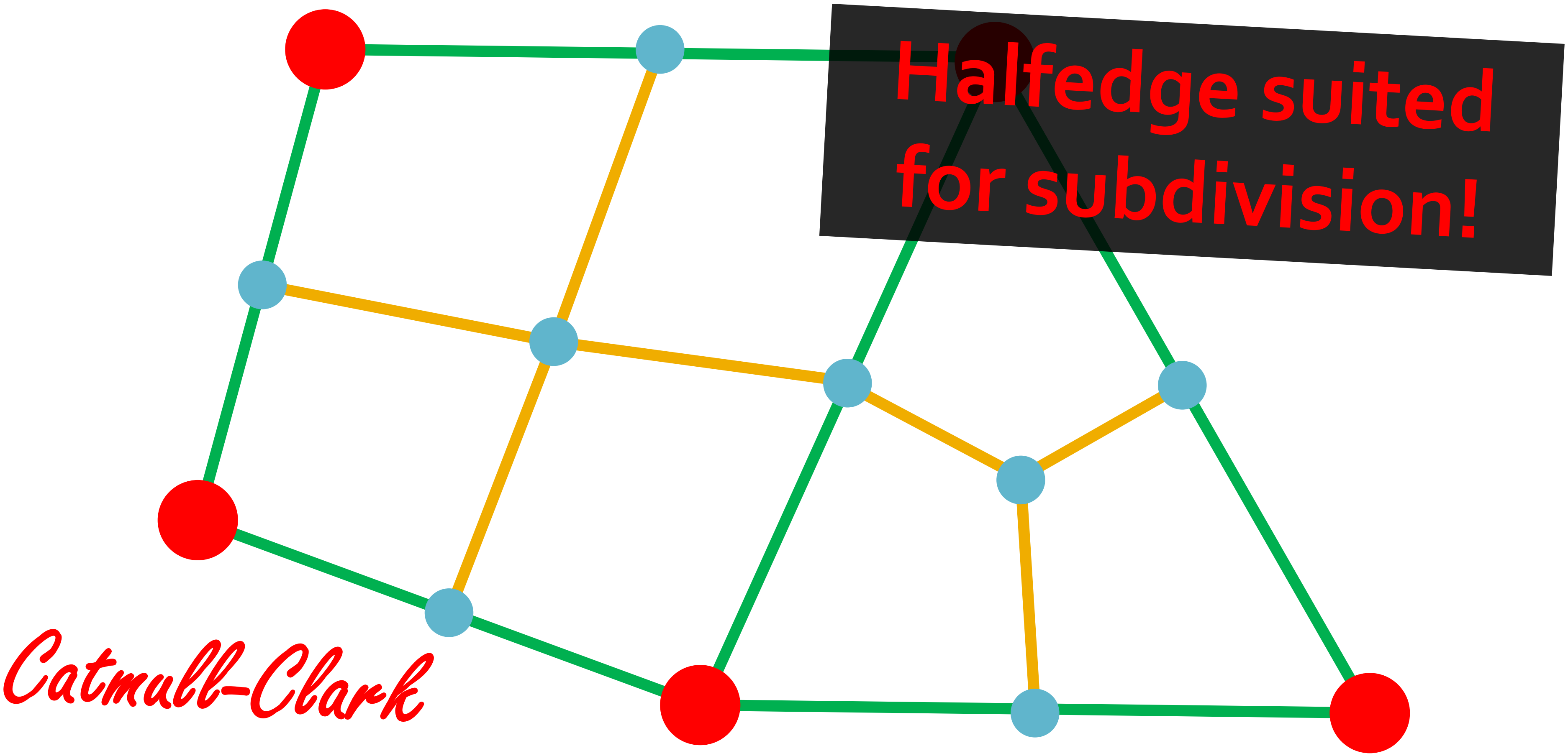


May have to convert to mesh

http://www.cad-sourcing.com/wp-content/uploads/2011/12/free-cad-software.jpg

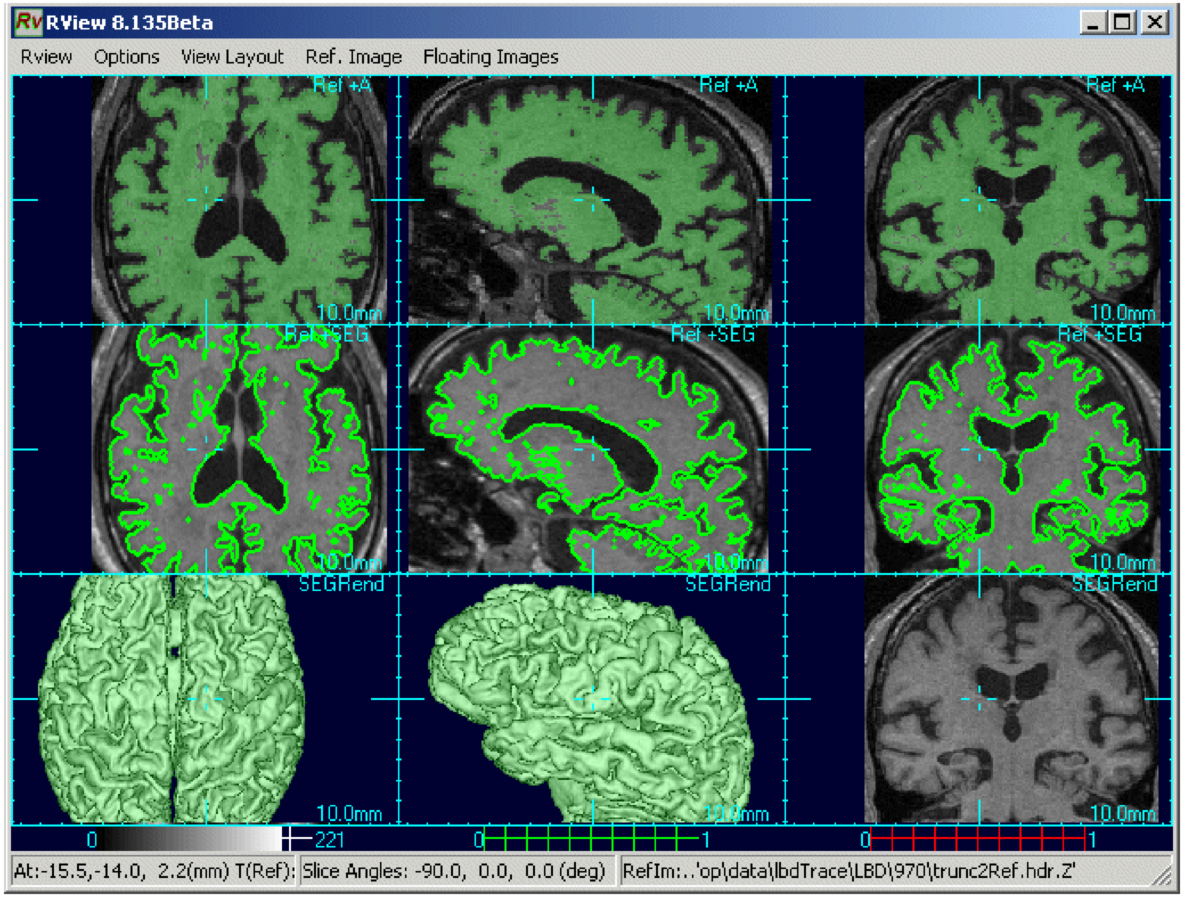
Cleanest: Design software

Obtaining Geometry



Cleanest: Design software

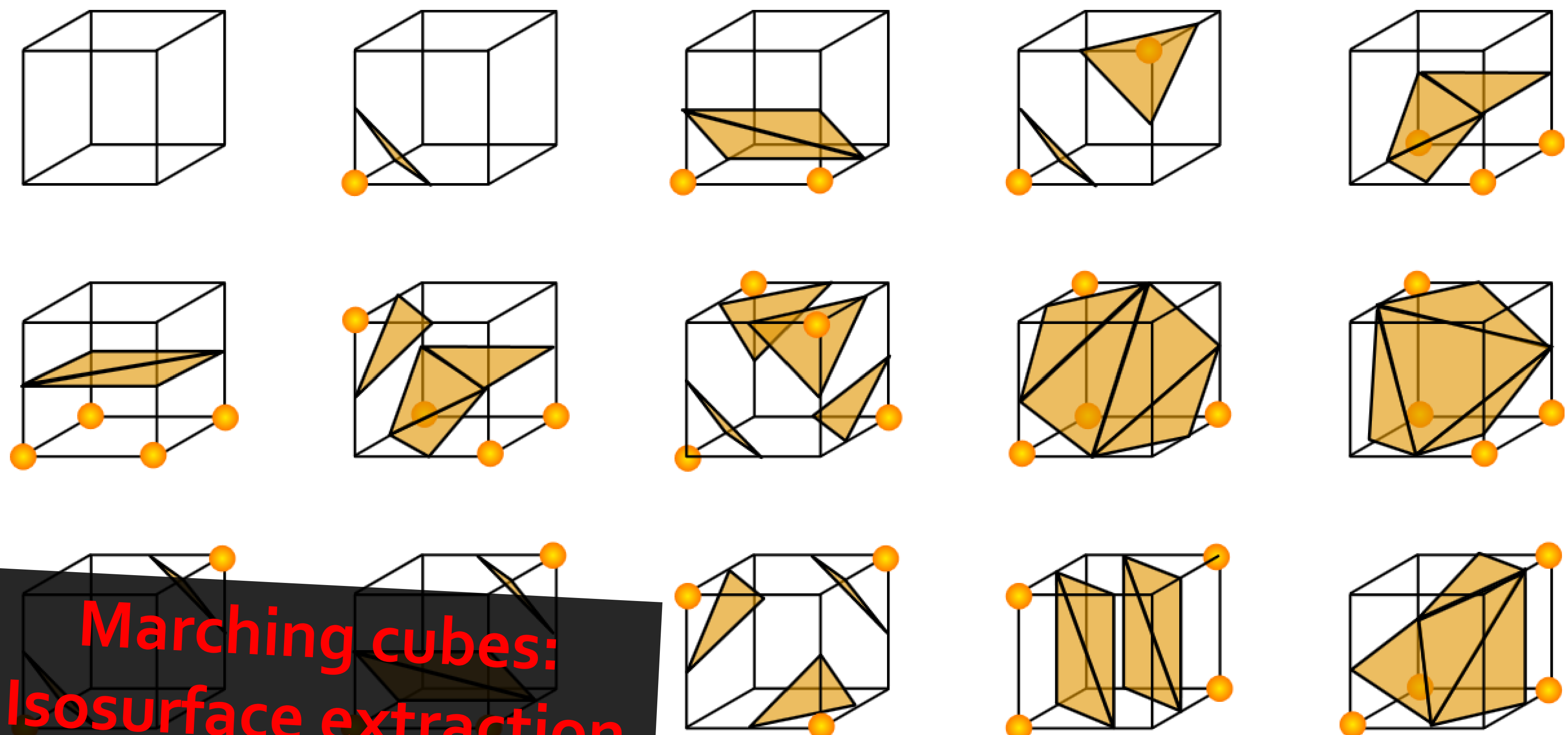
Obtaining Geometry



http://www.colin-studholme.net/software/rview/rvmanual/morphtool5.gif

Volumetric extraction

Obtaining Geometry

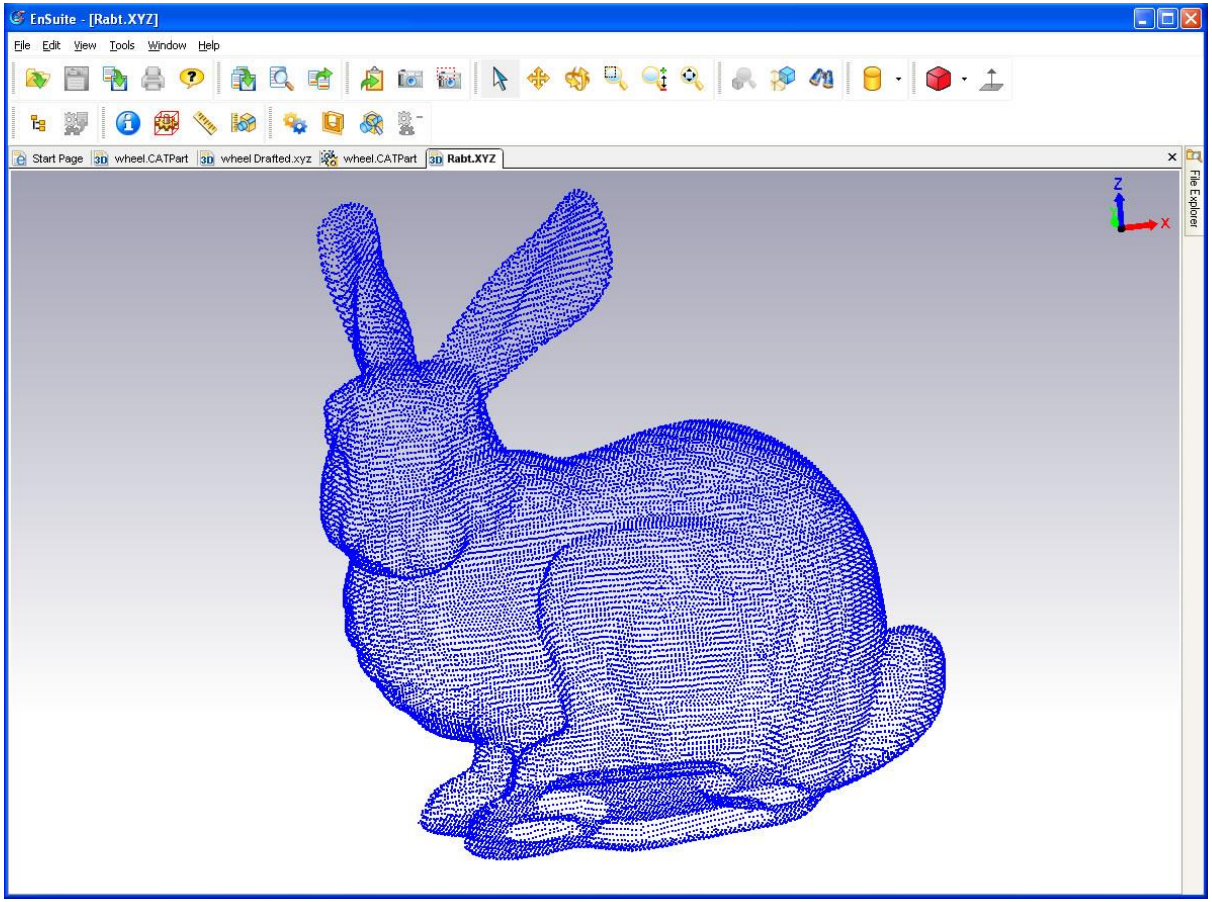


Marching cubes:
Isosurface extraction

http://en.wikipedia.org/wiki/Marching_cubes

Volumetric extraction

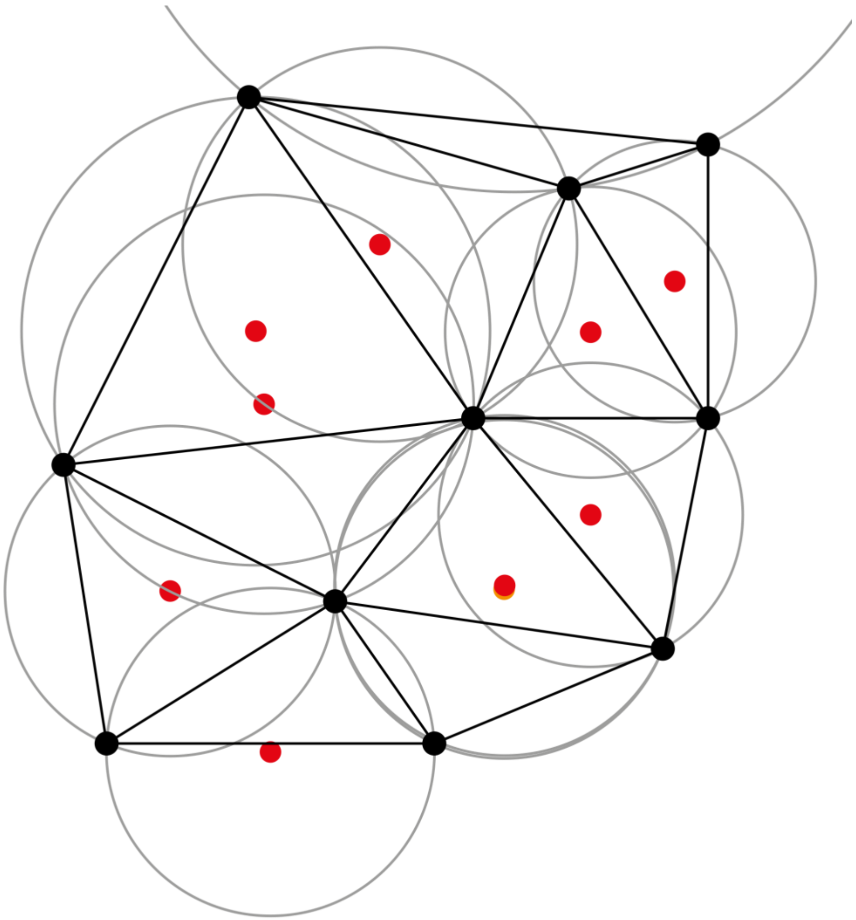
Obtaining Geometry

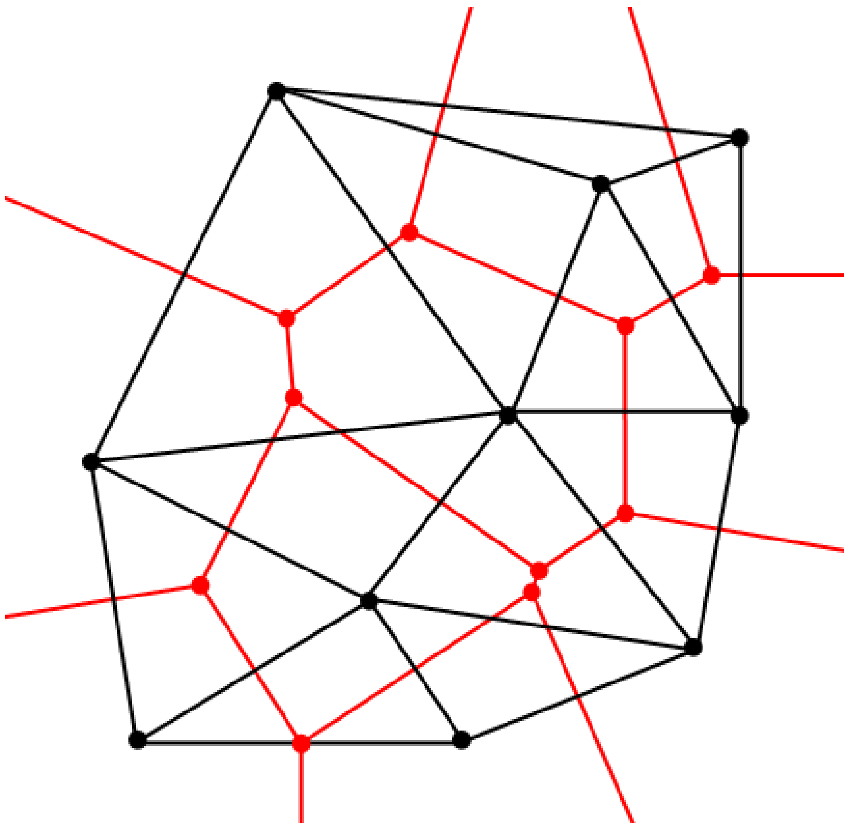


http://www.engineeringspecifier.com/public/primages/pr1200.jpg

Point clouds

Delaunay Triangulation





http://en.wikipedia.org/wiki/Delaunay_triangulation

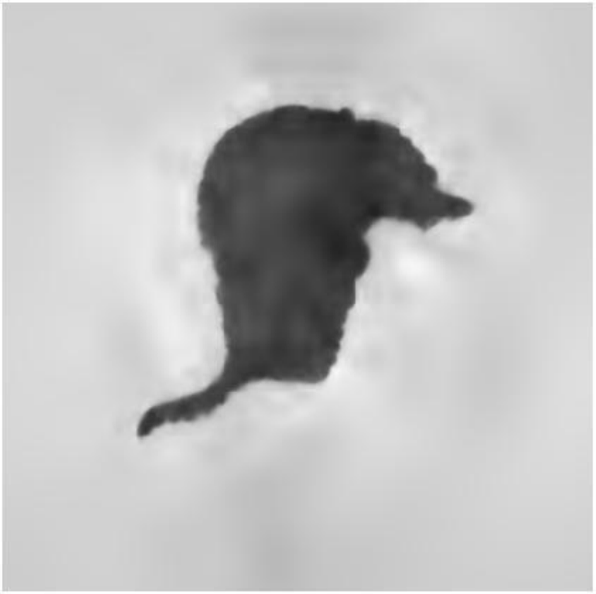
Well-behaved dual mesh

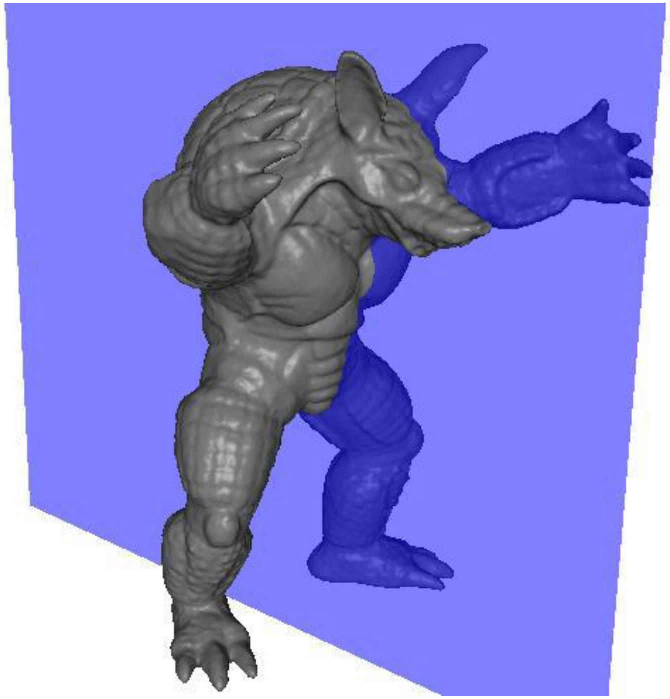
Strategies for Surface Delaunay

- **Tangent plane**
  Derive local triangulation from tangent projection
- **Restricted Delaunay**
  Usual Delaunay strategy but in smaller part of R^3
- **Inside/outside labeling**
  Find inside/outside labels for tetrahedra
- **Empty balls**
  Require existence of sphere around triangle with no other point

Delaunay Triangulation Based Surface Reconstruction: Ideas and Algorithms
Cazals and Giesen 2004

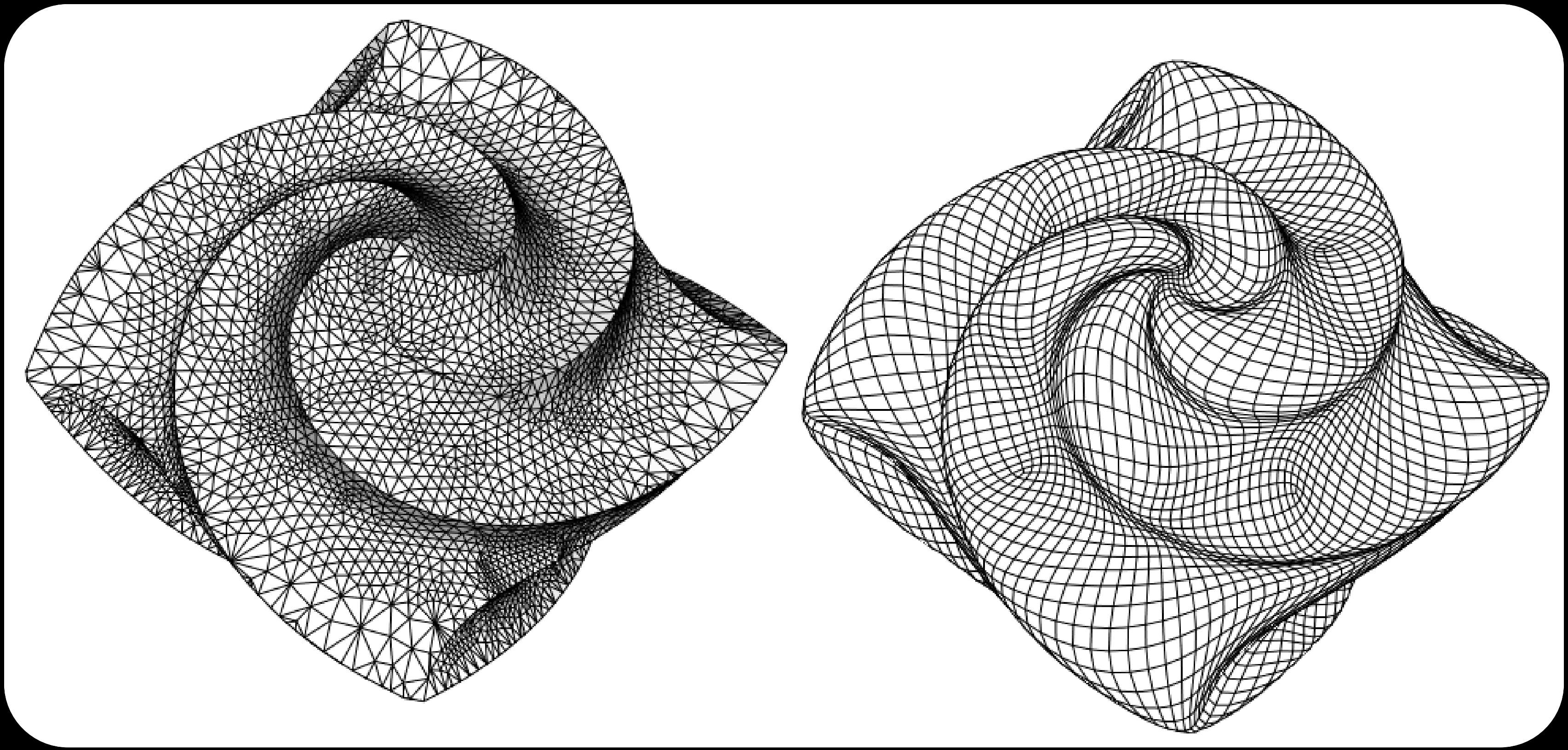
Poisson Reconstruction





Poisson Surface Reconstruction
Kazhdan, Bolitho and Hoppe (SGP 2006)

http://alice.loria.fr/index.php/publications.html?redirect=0&Paper=VSDM@2011&Author=levy



Representing Surfaces

Justin Solomon

MIT, Spring 2019