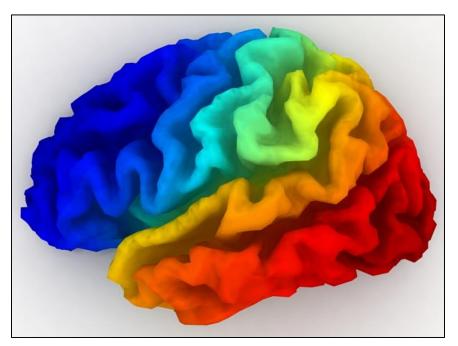


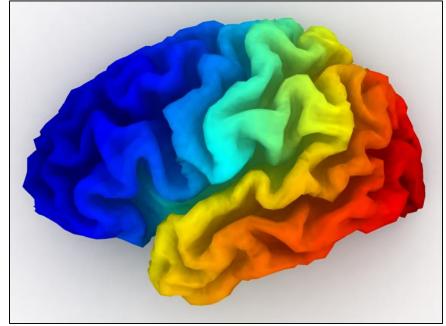
# **Surface Correspondence**

Justin Solomon
MIT, Spring 2019



# **Correspondence Problems**





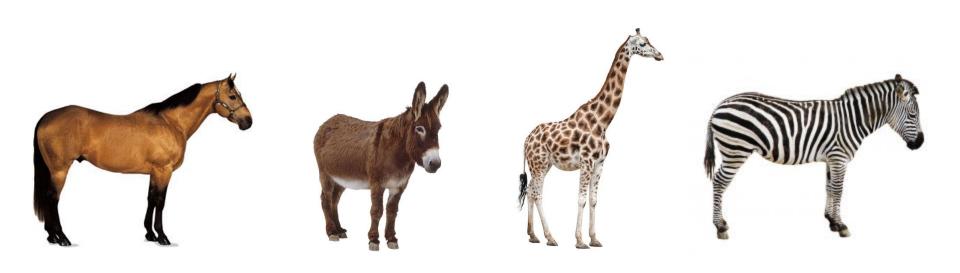
Which points on one object correspond to points on another?

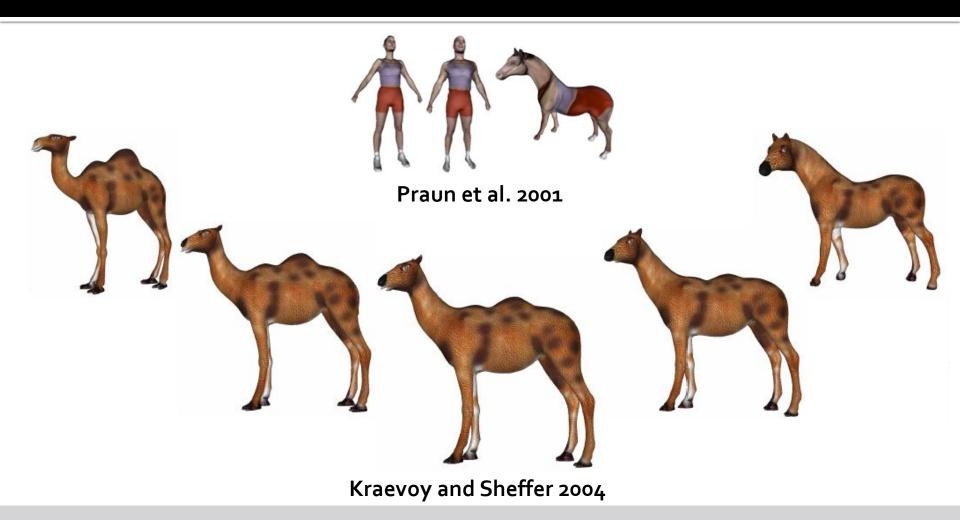


# How is this different from registration?

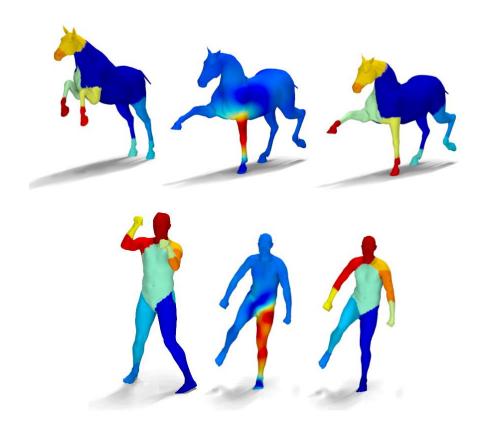
# **Typical Distinction**

# Seek shared structure instead of alignment



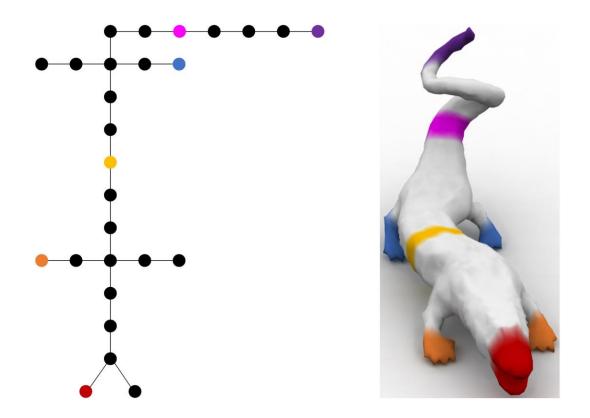


## **Texture transfer**



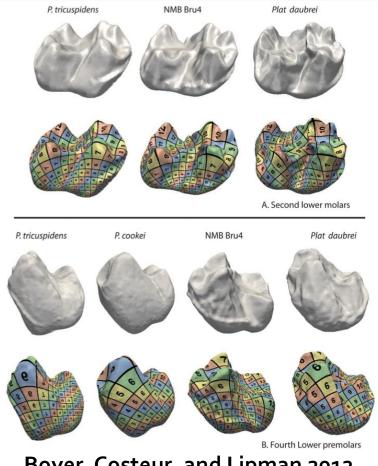
Ovsjanikov et al. 2012

# Segmentation transfer



Solomon et al. 2016

Layout



Boyer, Costeur, and Lipman 2012

# **Paleontology**

# **Desirable Properties**

### Given two (or more) shapes Find a map f, that is:

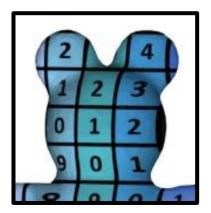
- Automatic
- Fast to compute
  - Bijective

(if we expect global correspondence)

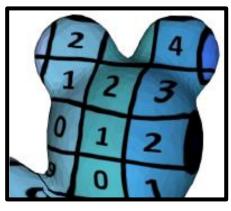
Low-distortion

What do we need the map for?

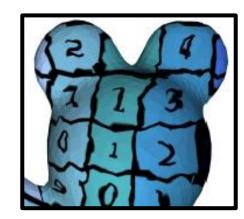
Shape interpolation or texture transfer require highly accurate map



Target Texture (projection)

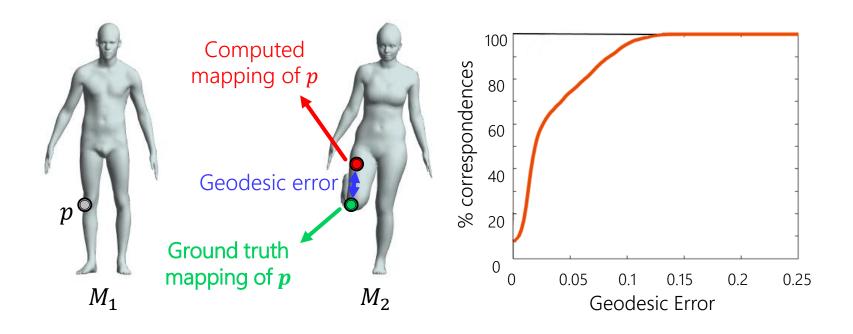


Locally and globally accurate map

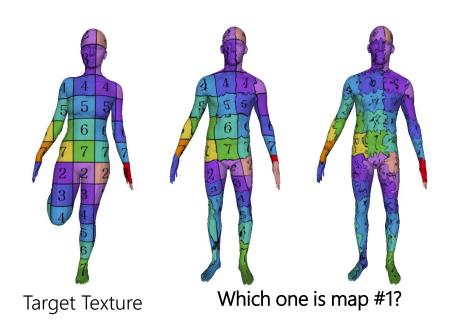


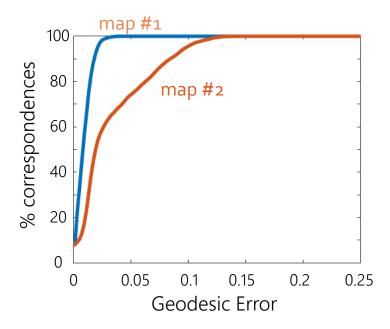
Globally accurate, locally distorted map

How can we evaluate a given map quality?
Given a ground truth map, compute the cumulative error graph

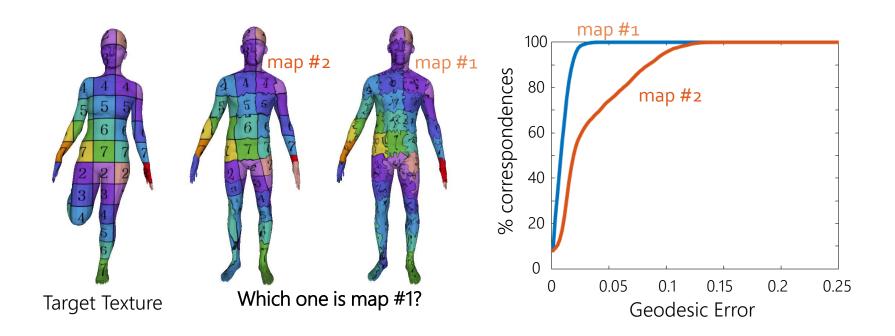


How can we evaluate a given map quality?
Given a ground truth map, compute the cumulative error graph



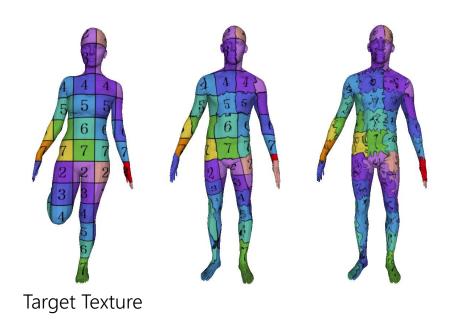


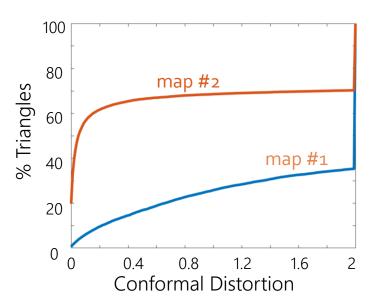
How can we evaluate a given map quality?
Given a ground truth map, compute the cumulative error graph



How can we evaluate a given map quality?

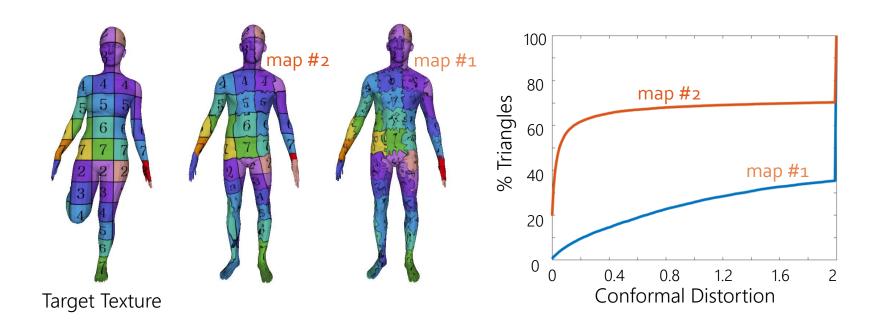
Measure *conformal distortion* (angle preservation)



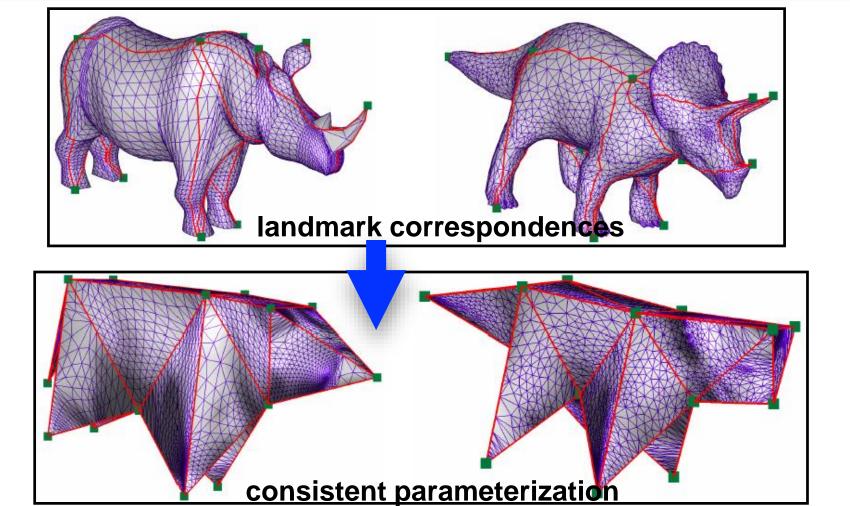


How can we evaluate a given map quality?

Measure *conformal distortion* (angle preservation)



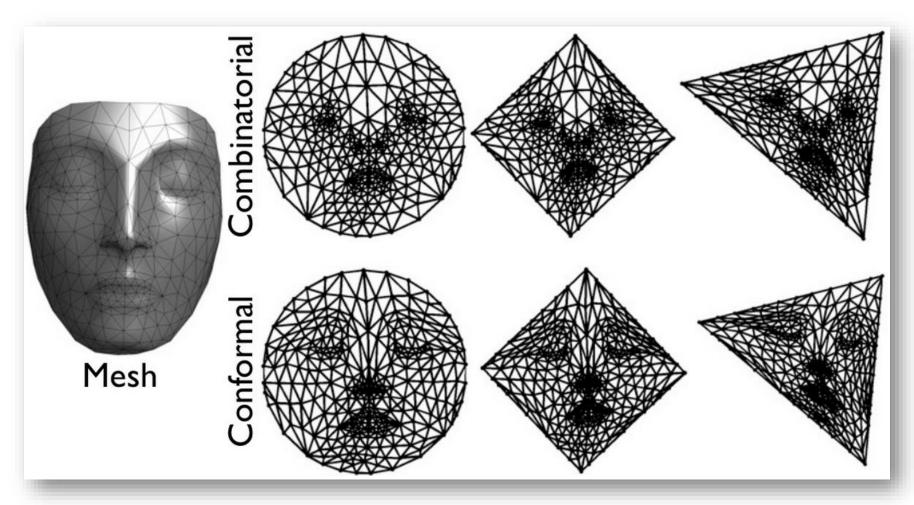
# **Example: Consistent Remeshing**



Kraevoy 2004



# Example: Mesh Embedding



G. Peyré, mesh processing course slides



# Linear Solve for Embedding

$$\min_{\mathbf{x}_1, \dots, \mathbf{x}_{|V|}} \quad \sum_{(i,j) \in E} w_{ij} \|\mathbf{x}_i - \mathbf{x}_j\|_2^2$$
  
s.t.  $\mathbf{x}_v \text{ fixed } \forall v \in V_0$ 

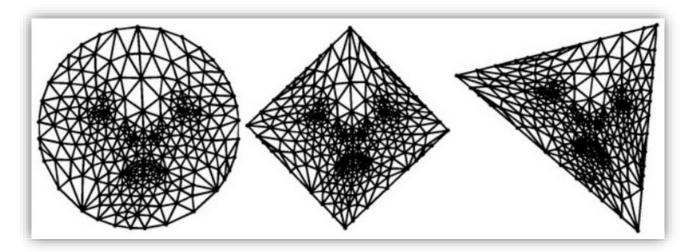
- $w_{ij} \equiv 1$ : Tutte embedding
- w<sub>ij</sub> from mesh: Harmonic embedding

Assumption: w symmetric.

# **Tutte Embedding Theorem**

$$\min_{\mathbf{x}_1, \dots, \mathbf{x}_{|V|}} \quad \sum_{(i,j) \in E} w_{ij} \|\mathbf{x}_i - \mathbf{x}_j\|_2^2$$
  
s.t.  $\mathbf{x}_v \text{ fixed } \forall v \in V_0$ 

Tutte embedding bijective if w nonnegative and boundary mapped to a convex polygon.

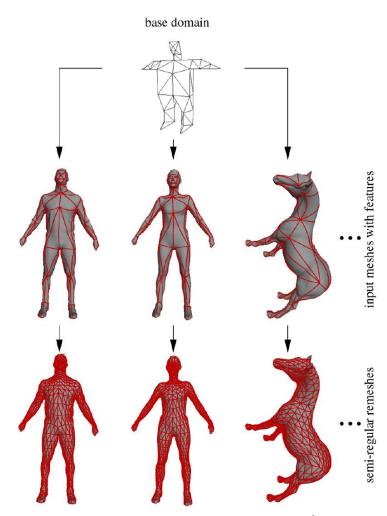


"How to draw a graph" (Proc. London Mathematical Society; Tutte, 1963)

# **Tradeoff: Consistent Remeshing**

#### Pros:

- Easy
- Straightforward applications
- Cons:
  - Need manual landmarks
  - Hard to minimize distortion



# Recently Revisited



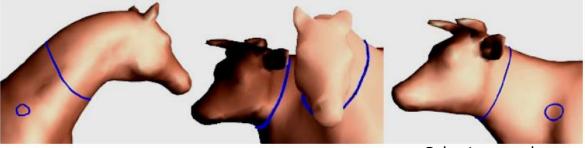
"Orbifold Tutte Embeddings" (Aigerman and Lipman, SIGGRAPH Asia 2015)

### **Automatic Landmarks**

- Simple algorithm:
  - Set landmarks
  - Measure energy
  - Repeat

- Possible metrics
  - Conformality
  - Area preservation
  - Stretch

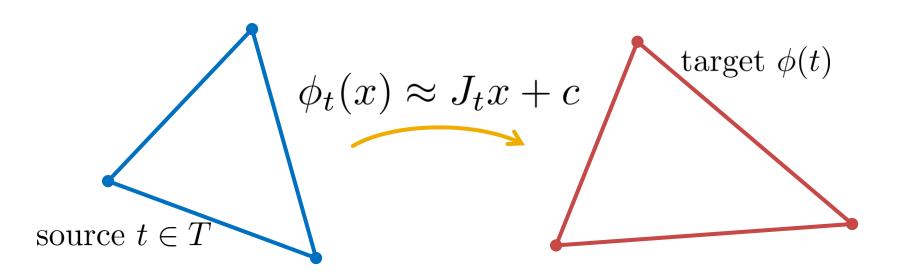




Schreiner et al. 2004

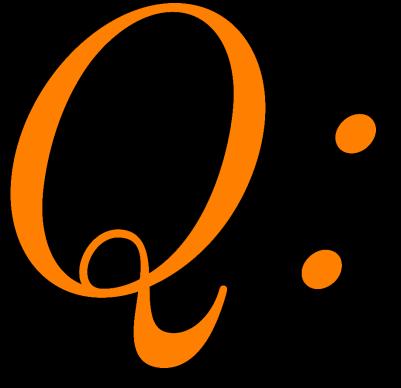


### **Local Distortion Measure**



Distortion := 
$$\sum_{t \in T} A_t \mathcal{D}(J_t)$$

Triangle distortion measure



# How do you measure distortion of a triangle?

# **Typical Distortion Measures**

Name	$\mathfrak{D}(\mathbf{J})$	$\mathcal{D}(\sigma)$
Symmetric Dirichlet	$\ \mathbf{J}\ _F^2 + \ \mathbf{J}^{-1}\ _F^2$	$\sum_{i=1}^{n} (\sigma_i^2 + \sigma_i^{-2})$
Exponential		
Symmetric	_	
Dirichlet	$\exp(s(\ \mathbf{J}\ _F^2 + \ \mathbf{J}^{-1}\ _F^2))$	$\exp(s\sum_{i=1}^{n}(\sigma_i^2+\sigma_i^{-2}))$
Hencky strain	$\left\ \log \mathbf{J}^{\! op}\!\mathbf{J} ight\ _F^2$	$\sum_{i=1}^{n} (\log^2 \sigma_i)$
AMIPS	$\exp(s \cdot \frac{1}{2} (\frac{\operatorname{tr}(\mathbf{J}^{T} \mathbf{J})}{\det(\mathbf{J})}$	$\exp(s(\frac{1}{2}(\frac{\sigma_1}{\sigma_2} + \frac{\sigma_2}{\sigma_1})$
	$+\frac{1}{2}(\det(\mathbf{J}) + \det(\mathbf{J}^{-1})))$	$+\frac{1}{4}(\sigma_1\sigma_2+\frac{1}{\sigma_1\sigma_2}))$
Conformal AMIPS 21	$D rac{\mathrm{tr}(\mathbf{J}^{ op}\mathbf{J})}{\det(\mathbf{J})}$	$\frac{\sigma_1^2 + \sigma_2^2}{\sigma_1 \sigma_2}$
Conformal AMIPS 3	$D\frac{\operatorname{tr}(\mathbf{J}^{T}\mathbf{J})}{\det(\mathbf{J})^{\frac{2}{3}}}$	$rac{\sigma_1\sigma_2}{\sigma_1^2+\sigma_2^2+\sigma_3^2}$ Open challenge $(\sigma_1\sigma_2\sigma_3)^{rac{2}{3}}$ Optimize
		directly

Table from "Scalable Locally Injective Mappings" (Rabinovich et al., 2017)

## Related Problem

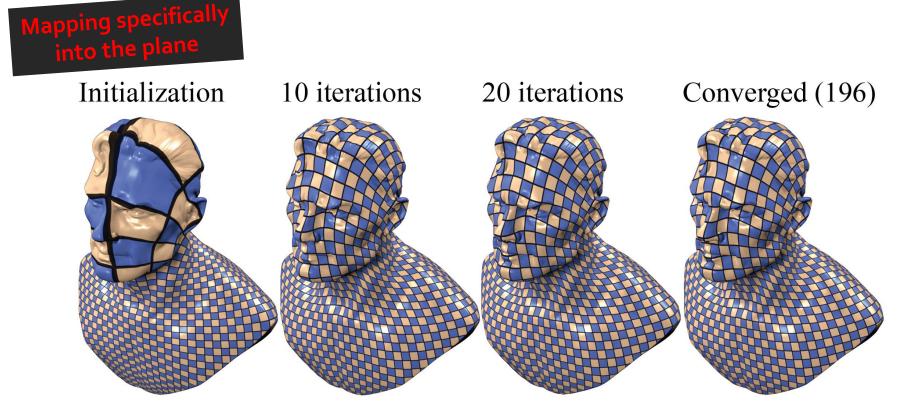


Image from "Scalable Locally Injective Mappings" (Rabinovich et al., 2017)

### **Parameterization**

### **New Idea**

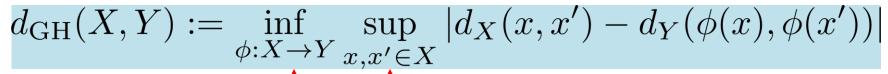
Not all calculations have to be at the triangle level!

# Long-distance interactions

can stabilize geometric computations.

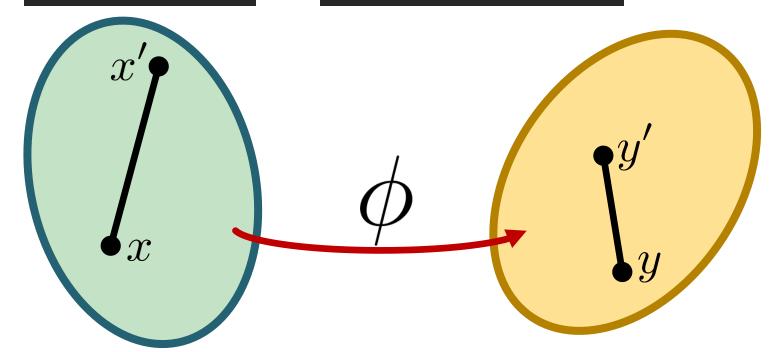
### **Gromov-Hausdorff Distance**

#### Distance between metric spaces X, Y





#### **Worst distortion**



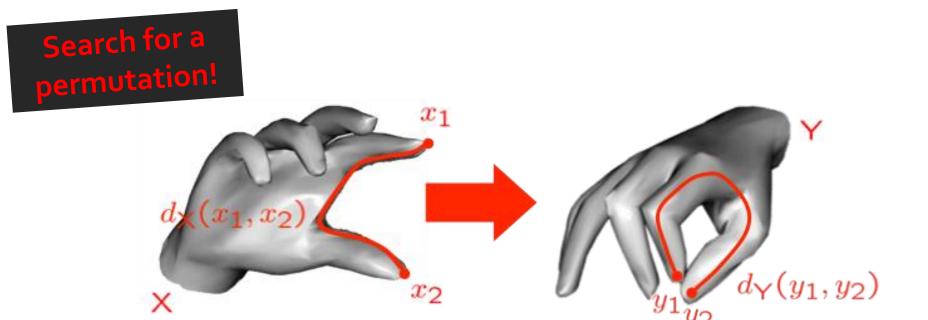
# Classical Multidimensional Scaling

- 1. Double centering:  $B := -\frac{1}{2}JDJ$ Centering matrix  $J := I - \frac{1}{n}\mathbf{1}\mathbf{1}^{\top}$
- 2. Find m largest eigenvalues/eigenvectors

3. 
$$X = E_m \Lambda_m^{1/2}$$



### **Generalized MDS**



$$d_{\text{int}}(X,Y) := \min_{\{y_1, \dots, y_n\} \subset Y} \|d_X(x_i, x_j) - d_Y(y_i, y_j)\|$$

# Problem: Quadratic Assignment

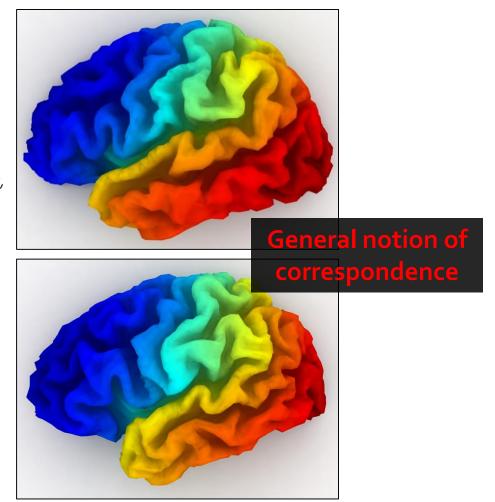
$$\min_{T} \quad \langle M_0 T, T M_1 \rangle$$
s.t.  $T \in \{0, 1\}^{n \times n}$ 

$$T \mathbf{1} = p_0$$

$$T^{\mathsf{T}} \mathbf{1} = p_1$$

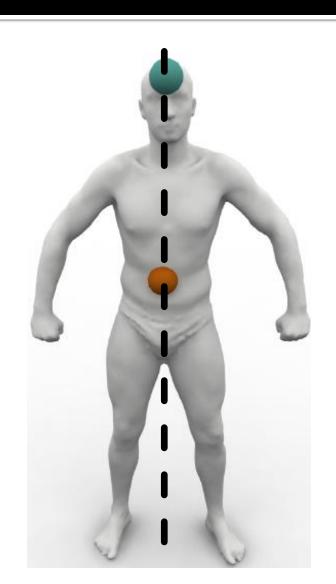
Nonconvex quadratic program!

NP-hard!



# What's Wrong?

- Hard to optimize
- Multiple optima



### **Tradeoff: GMDS**

- Pros:
  - Good distance for non-isometric metric spaces
- Cons:
  - Non-convex
  - HUGE search space (i.e. permutations)

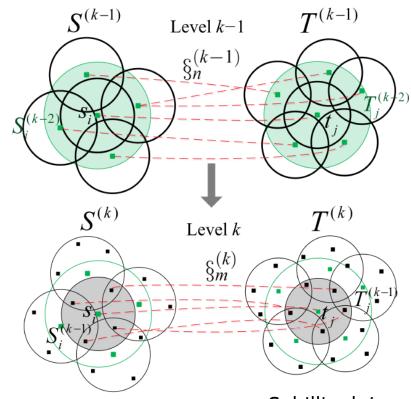
## **GMDS** in Practice

- Heuristics to explore the permutations
  - Solve at a very coarse scale and interpolate
  - Coarse-to-fine
  - Partial matching



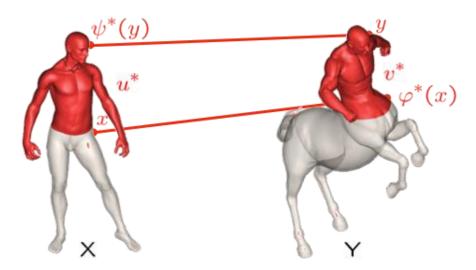
### **GMDS** in Practice

- Heuristics to explore the permutations
  - Solve at a very coarse scale and interpolate
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  - Partial matching



### **GMDS** in Practice

- Heuristics to explore the permutations
  - Solve at a very coarse scale and interpolate
  - Coarse-to-fine
  - Partial matching





- Find correspondence  $\varphi^*, \psi^*$  minimizing distortion between current parts  $u^*, v^*$
- Select parts  $u^*, v^*$  minimizing the distortion with current correspondence  $\varphi^*, \psi^*$  subject to  $\lambda(u^*, v^*) \leq \lambda_0$

### Returning to Desirable Properties

#### Given two (or more) shapes Find a map f, that is:

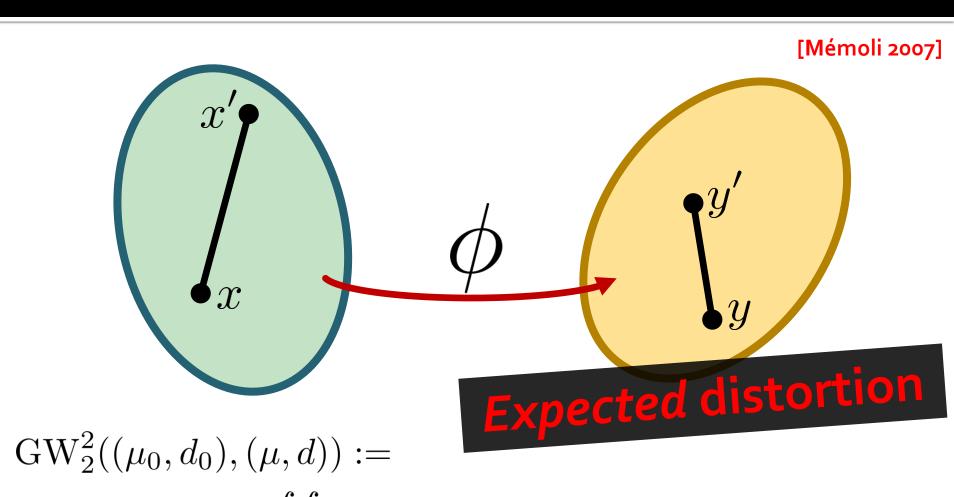
- Automatic
- Fast to compute
  - Bijective

(if we expect global correspondence)

Low-distortion

(unless local optimum is bad)

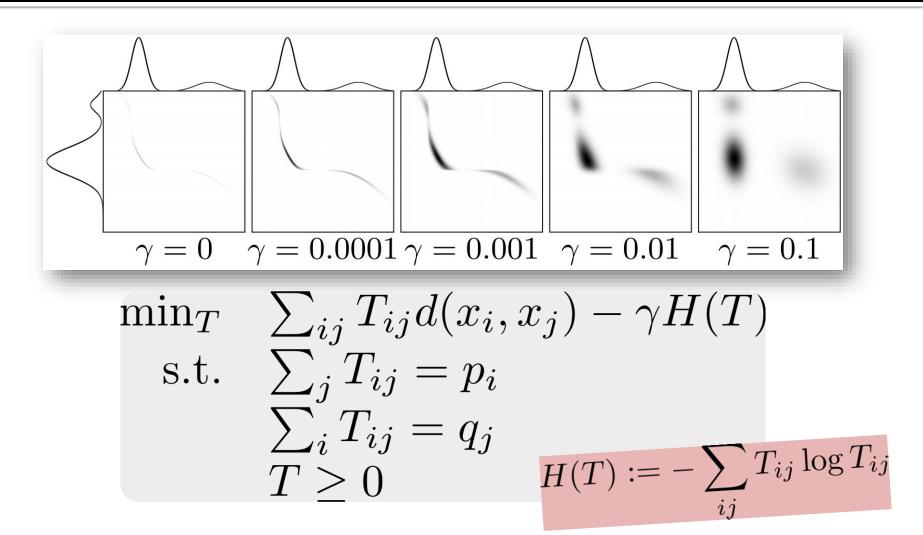
#### **Gromov-Wasserstein Distance**



$$\min_{\gamma \in \mathcal{M}(\mu_0, \mu)} \iint_{\Sigma_0 \times \Sigma} [d_0(x, x') - d(y, y')]^2 d\gamma(x, y) d\gamma(x', y')$$



### **Entropic Regularization**



Cuturi. "Sinkhorn distances: Lightspeed computation of optimal transport" (NIPS 2013)

### **Gromov-Wasserstein Plus Entropy**

#### **Entropic Metric Alignment for Correspondence Problems**

Justin Solomon\* MIT

Gabriel Peyré CNRS & Univ. Paris-Dauphine Vladimir G. Kim Adobe Research Suvrit Sra MIT

#### Abstract

Many shape and image processing tools rely on computation of correspondences between geometric domains. Efficient methods that stably extract "soft" matches in the presence of diverse geometric structures have proven to be valuable for shape retrieval and transfer of labels or semantic information. With these applications in mind, we present an algorithm for probabilistic correspondence that optimizes an entropy-regularized Gromov-Wasserstein (GW) objective. Built upon recent developments in numerical optimal transportation, our algorithm is compact, provably convergent, and applicable to any geometric domain expressible as a metric measure matrix. We provide comprehensive experiments illustrating the convergence and applicability of our algorithm to a variety of graphics tasks. Furthermore, we expand entropic GW correspondence to a framework for other matching problems, incorporating partial distance matrices, user guidance, shape exploration, symmetry detection, and joint analysis of more than two domains. These applications expand the scope of entropic GW correspondence to major shape analysis problems and are stable to distortion and noise.

**Keywords:** Gromov-Wasserstein, matching, entropy

Concepts: •Computing methodologies → Shape analysis;

#### Introduction

A basic component of the geometry processing toolbox is a tool for mapping or correspondence, the problem of finding which points on a target domain correspond to points on a source. Many variations of this problem have been considered in the graphics literature, e.g.

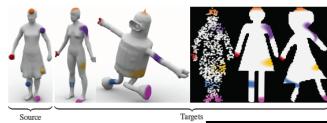


Figure 1: Entropic GW can find correspond surface (left) and a surface with similar shared semantic structure, a noisy 3D po hand drawing. Each fuzzy map was comp

are violated these algorithms suffer from local elastic terms into a single global ma

In this paper, we propose a new corres minimizes distortion of long- and shortstudy an entropically-regularized version of (GW) mapping objective function from [ the distortion of geodesic distances. The o matching expressed as a "fuzzy" correspon of [Kim et al. 2012; Solomon et al. 2012] the correspondence via the weight of an e

Although [Mémoli 2011] and subsequent bility of using GW distances for geometric tional challenges hampered their practical these challenges, we build upon recent me timal transportation introduced in [Benan et al. 2015]. While optimal transportation ent ontimization problem from regularized GW computation (linear

function Gromov-Wasserstein $(\mu_0, \mathbf{D}_0, \mu, \mathbf{D}, \alpha, \eta)$ // Computes a local minimizer  $\Gamma$  of (6)  $\Gamma \leftarrow \text{ONES}(n_0 \times n)$ for  $i = 1, 2, 3, \dots$  $\mathbf{K} \leftarrow \exp(\mathbf{D}_0 \llbracket \boldsymbol{\mu}_0 \rrbracket \boldsymbol{\Gamma} \llbracket \boldsymbol{\mu} \rrbracket \mathbf{D}^\top / \alpha)$  $\Gamma \leftarrow \text{SINKHORN-PROJECTION}(\mathbf{K}^{\wedge \eta} \otimes \mathbf{\Gamma}^{\wedge (1-\eta)}; \boldsymbol{\mu}_0, \boldsymbol{\mu})$ 

**function** SINKHORN-PROJECTION( $\mathbf{K}; \boldsymbol{\mu}_0, \boldsymbol{\mu}$ ) // Finds  $\Gamma$  minimizing  $\mathrm{KL}(\Gamma|\mathbf{K})$  subject to  $\Gamma \in \overline{\mathcal{M}}(\mu_0, \mu)$  $\mathbf{v}, \mathbf{w} \leftarrow \mathbf{1}$ for  $j = 1, 2, 3, \dots$  $\mathbf{v} \leftarrow \mathbf{1} \oslash \mathbf{K}(\mathbf{w} \otimes \boldsymbol{\mu})$  $\mathbf{w} \leftarrow \mathbf{1} \oslash \mathbf{K}^{\top} (\mathbf{v} \otimes \boldsymbol{\mu}_0)$ return [v]K[w]

Algorithm 1: Iteration for finding regularized Gromov-Wasserstein distances.  $\otimes$ ,  $\otimes$  denote elementwise multiplication and division.

return  $\Gamma$ 

#### **Convex Relaxation**

#### **Tight Relaxation of Quadratic Matching**

Itay Kezurer<sup>†</sup> Shahar Z. Kovalsky<sup>†</sup> Ronen Basri Yaron Lipman

Weizmann Institute of Science

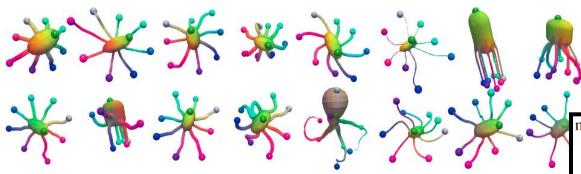


Figure 1: Consistent Collection Matching. Results of the proposed one-stage procedure for finding consistent correspondence in a collection showing strong variability and non-rigid deformations.

#### Abstract

Establishing point correspondences between shapes is extremely challenging as it involves both finding se semantically persistent feature points, as well as their combinatorial matching. We focus on the latter and consthe Quadratic Assignment Matching (QAM) model. We suggest a novel convex relaxation for this NP-hard protect that builds upon a rank-one reformulation of the problem in a higher dimension, followed by relaxation in semidefinite program (SDP). Our method is shown to be a certain hybrid of the popular spectral and does stochastic relaxations of QAM and in particular we prove that it is tighter than both.

Experimental evaluation shows that the proposed relaxation is extremely tight: in the majority of our experim it achieved the certified global optimum solution for the problem, while other relaxations tend to produce optimal solutions. This, however, comes at the price of solving an SDP in a higher dimension.

$$\max_{Y} \quad \operatorname{tr}(WY)$$
s.t.  $Y \succeq [X][X]^{T}$ 

$$X \in \operatorname{conv} \Pi_{n}^{k}$$

$$\operatorname{tr}Y = k$$

Y > 0

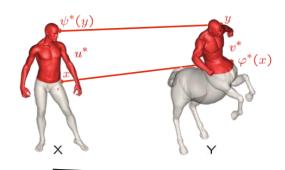
$$\sum_{qrst} Y_{qrst} = k^2$$

$$Y_{qrst} \le \begin{cases} 0, & \text{if } q = s, \ r \ne t \\ 0, & \text{if } r = t, \ q \ne s \\ \min\{X_{qr}, X_{st}\}, & \text{otherwise} \end{cases}$$

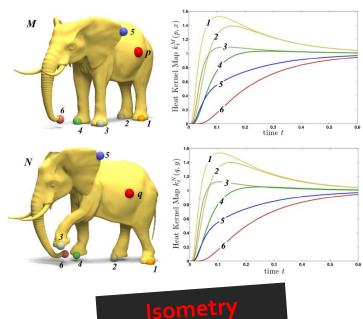
#### Continuum

#### Weak assumptions

#### **Strong assumptions**

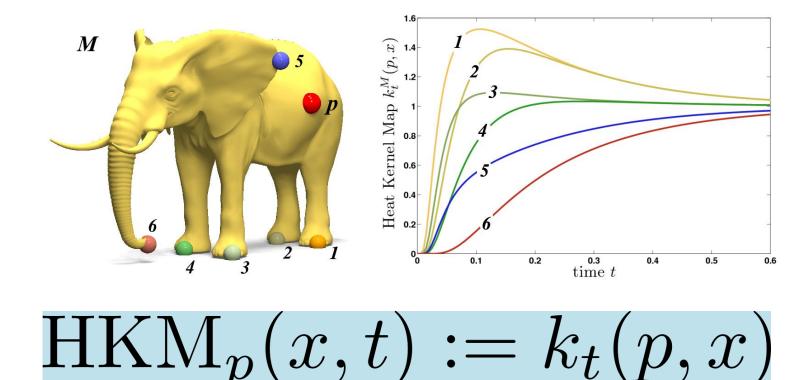


Low-distortion



Recall:

#### **Heat Kernel Map**



Theorem: Only have to match one point!



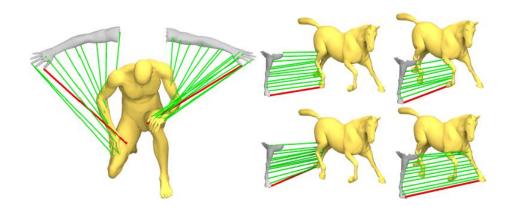
### Tradeoff: Heat Kernel Map

#### Pros:

- Tiny search space
- Some extension to partial matching

#### Cons:

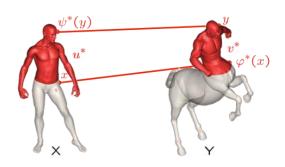
 (Extremely) sensitive to deviation from isometry





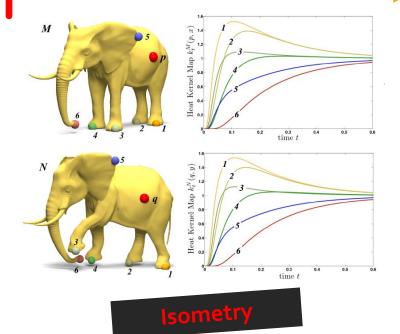
#### Continuum

#### Weak assumptions



Low-distortion

# **????**Strong assumptions



#### **Observation About Mapping**

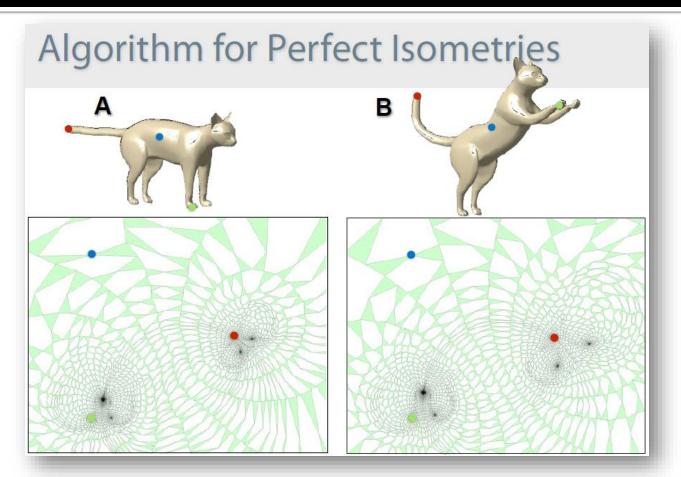
Angle and area preserving isometries  $\subseteq$  conformal maps

Easier

Hard!

Möbius Voting for Surface Correspondence Lipman and Funkhouser 2009

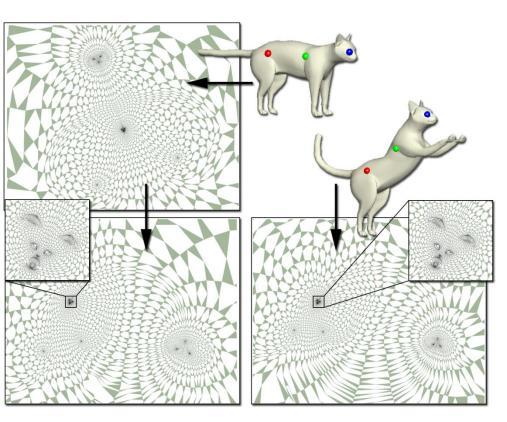
# O(n³) Algorithm for Perfect Isometry



 $http://www.mpi-inf.mpg.de/resources/deformableShapeMatching/EG2011\_Tutorial/slides/4.3\%20SymmetryApplications.pdf$ 

#### Map triplets of points

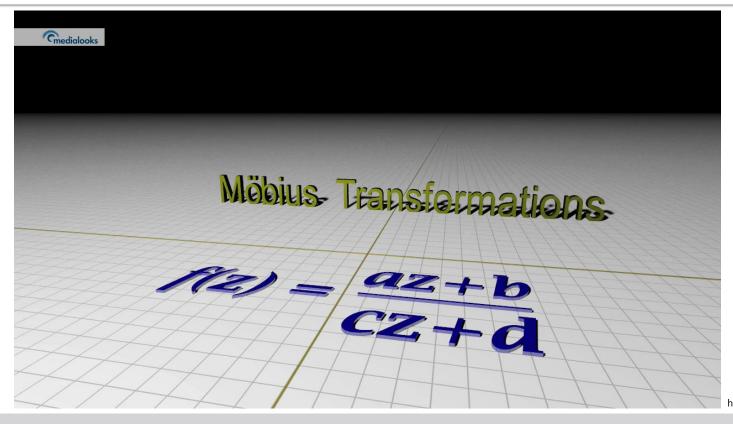
### Möbius Voting



- 1. Map surfaces to complex plane
- 2. Select three points
- 3. Map plane to itself matching these points
- 4. Vote for pairings using distortion metric to weight
- 5. Return to 2

Möbius Voting for Surface Correspondence Lipman and Funkhouser 2009

#### Möbius Transformations

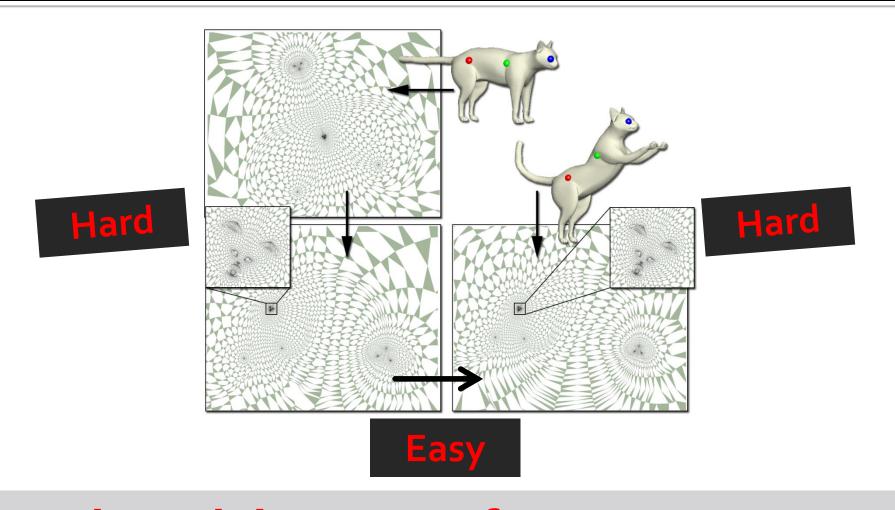


$$\frac{az+b}{cz+d}$$

http://www.ima.umn.edu/~arnold//moebius

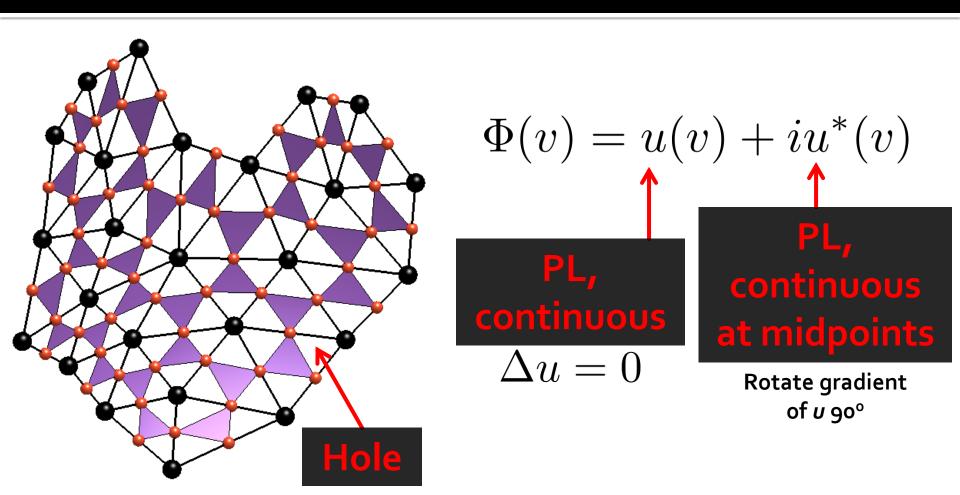
# Bijective conformal maps of the extended complex plane

#### Observation



Hard work is per-surface, not per-map

# Mid-Edge Uniformization



#### Cannot scale triangles to flatten

## **Voting Algorithm**

```
Input: points \Sigma_1 = \{z_k\} and \Sigma_2 = \{w_\ell\}
         number of iterations I
         minimal subset size K
Output: correspondence matrix C = (C_{k,\ell}).
/* Möbius voting
while number of iterations < I do
     Random z_1, z_2, z_3 \in \Sigma_1.
     Random w_1, w_2, w_3 \in \Sigma_2.
     Find the Möbius transformations m_1, m_2 s.t.
           m_1(z_i) = y_i, m_2(w_i) = y_i, j = 1, 2, 3.
     Apply m_1 on \Sigma_1 to get \bar{z}_k = m_1(z_k).
     Apply m_2 on \Sigma_2 to get \bar{w}_\ell = m_2(w_\ell).
     Find mutually nearest-neighbors (\bar{z}_k, \bar{w}_\ell) to formulate
     candidate correspondence c.
     if number of mutually closest pairs \geq K then
          Calculate the deformation energy \mathbf{E}(c)
          /* Vote in correspondence matrix
          foreach (\bar{z}_k, \bar{w}_\ell) mutually nearest-neighbors do
              C_{k,\ell} \leftarrow C_{k,\ell} + \frac{1}{\varepsilon + \mathbf{E}(c)/n}.
          end
     end
end
```

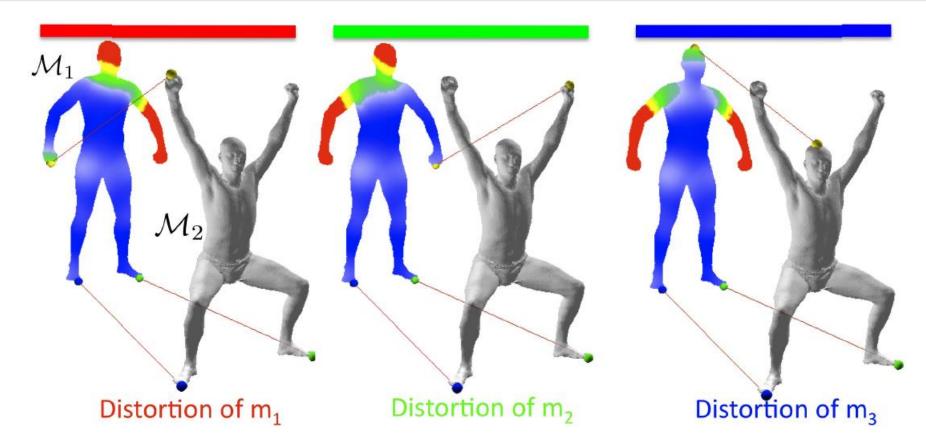
### Tradeoff: Möbius Voting

#### Pros:

- Efficient
- Voting procedure handles some non-isometry

#### Cons:

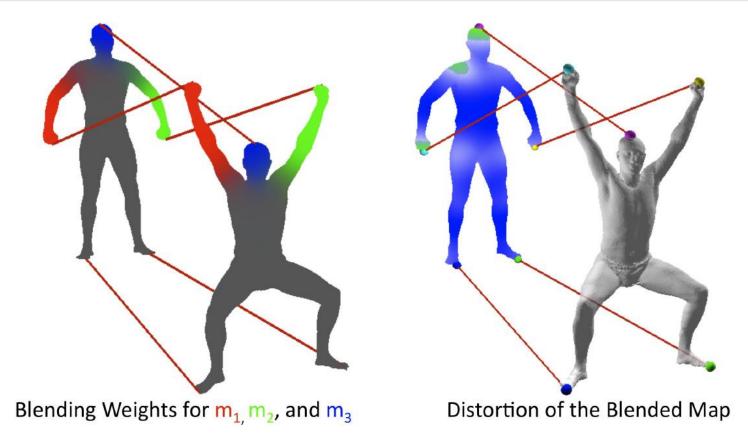
- Does not provide smooth/continuous map
- Does not optimize global distortion
- Only for genus o



Different conformal maps distorted in different places.

Blended Intrinsic Maps
Kim, Lipman, and Funkhouser 2011

### **Use for Dense Mapping**



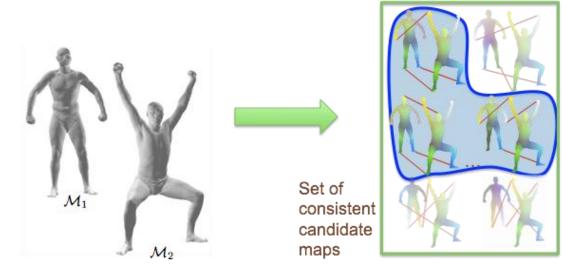
#### Combine good parts of different maps!

Blended Intrinsic Maps
Kim, Lipman, and Funkhouser 2011

- Algorithm:
  - Generate consistent maps
  - Find blending weights per-point on each map
  - Blend maps

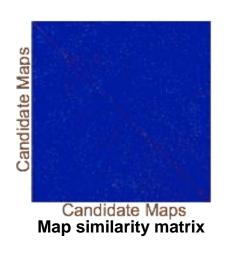
#### Algorithm:

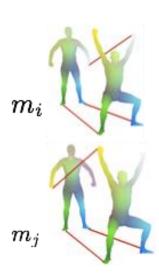
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#### Algorithm:

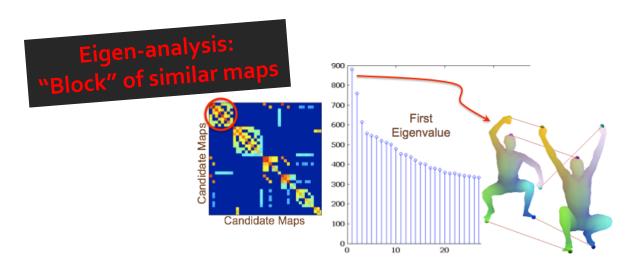
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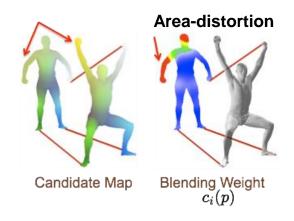


#### Algorithm:

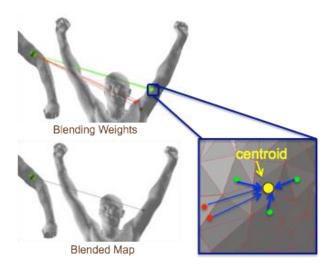
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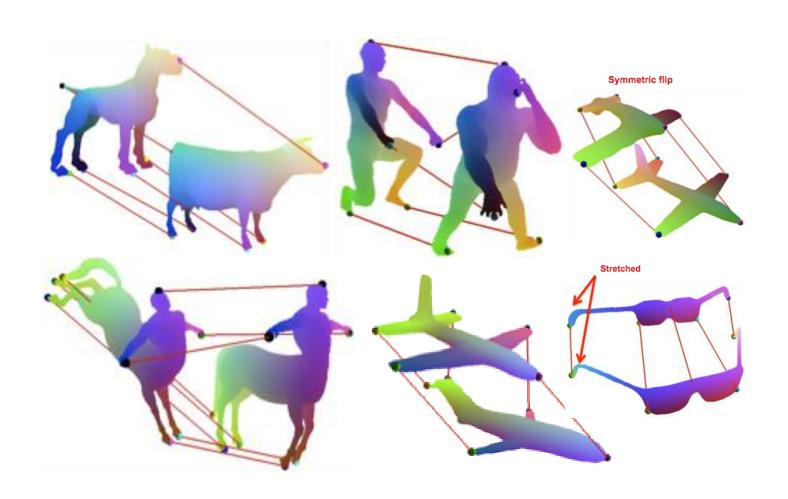
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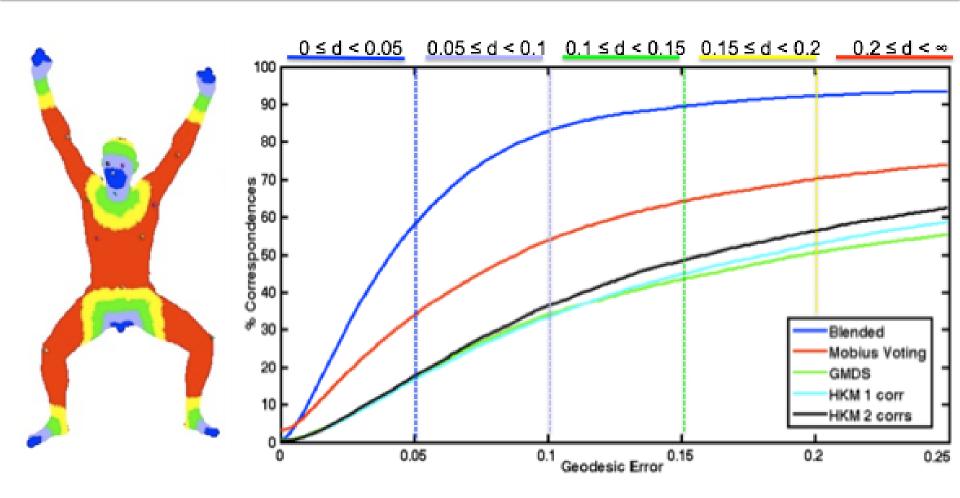
- Algorithm:
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### Some Examples



#### **Evaluation**



### Tradeoff: Blended Intrinsic Maps

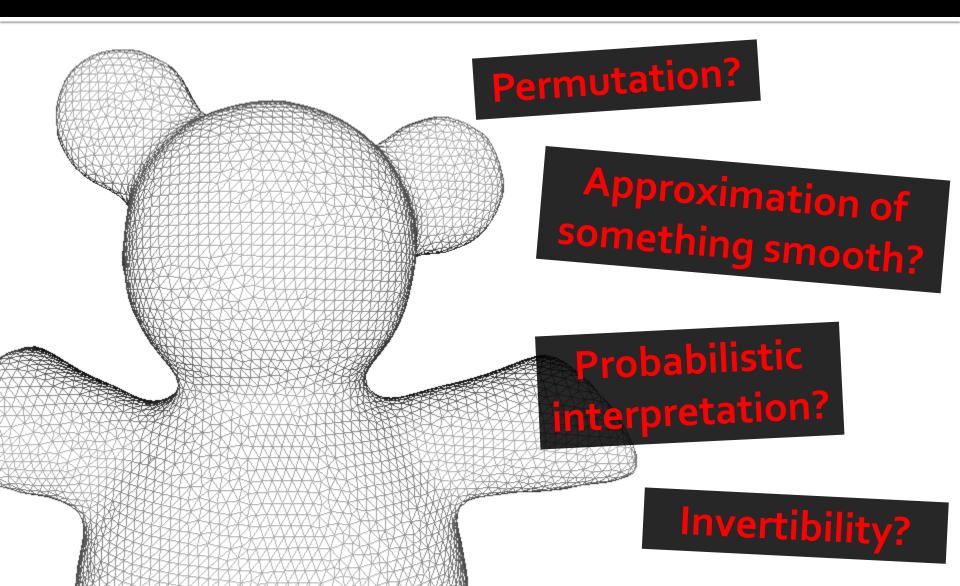
#### Pros:

- Can handle non-isometric shapes
- Efficient

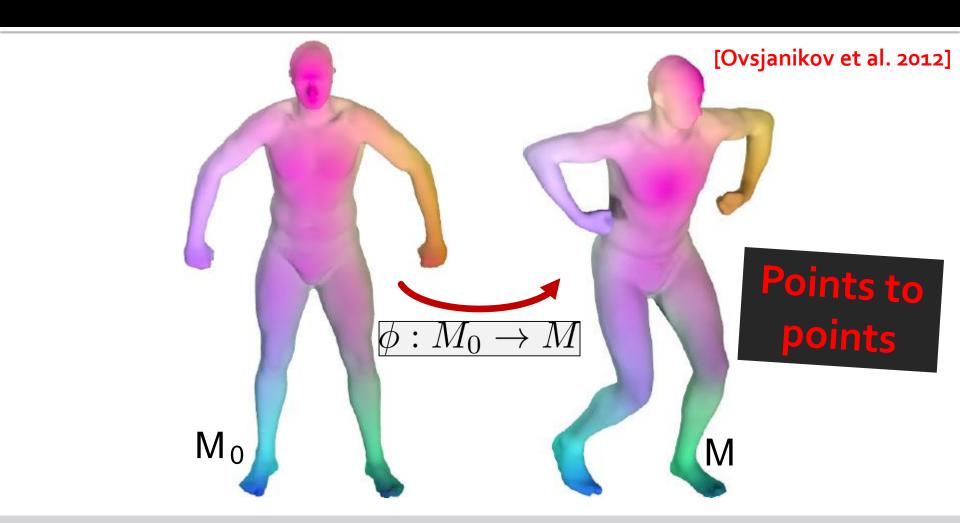
#### Cons:

- Lots of area distortion for some shapes
- Genus o manifold surfaces

### Subtlety: Representation

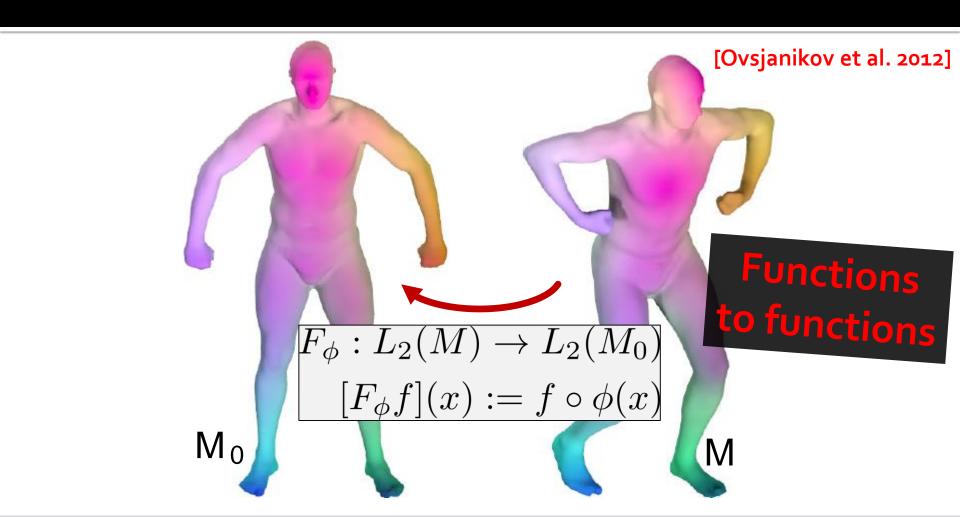


### **Functional Maps**



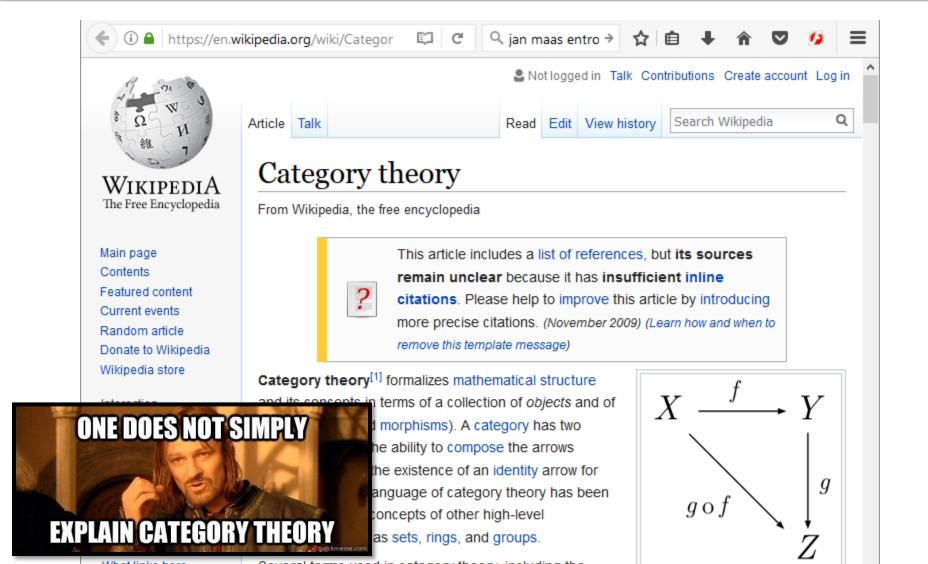
Points on  $M_o$  to points on M

#### **Functional Maps**



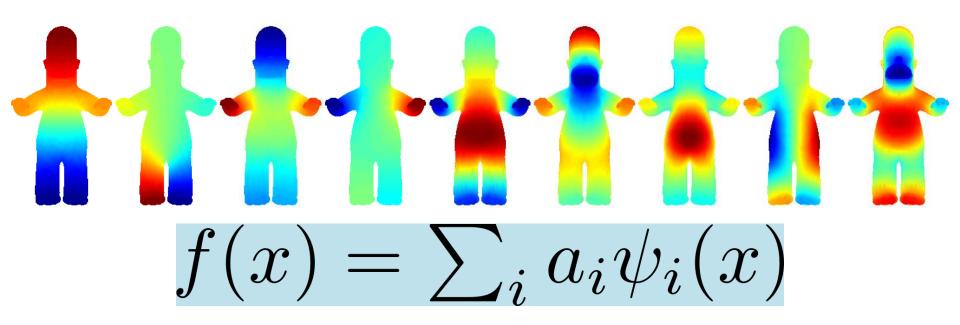
#### Functions on M to functions on $M_{\odot}$

#### **Mathematical Sidebar**



#### **Functional Maps**

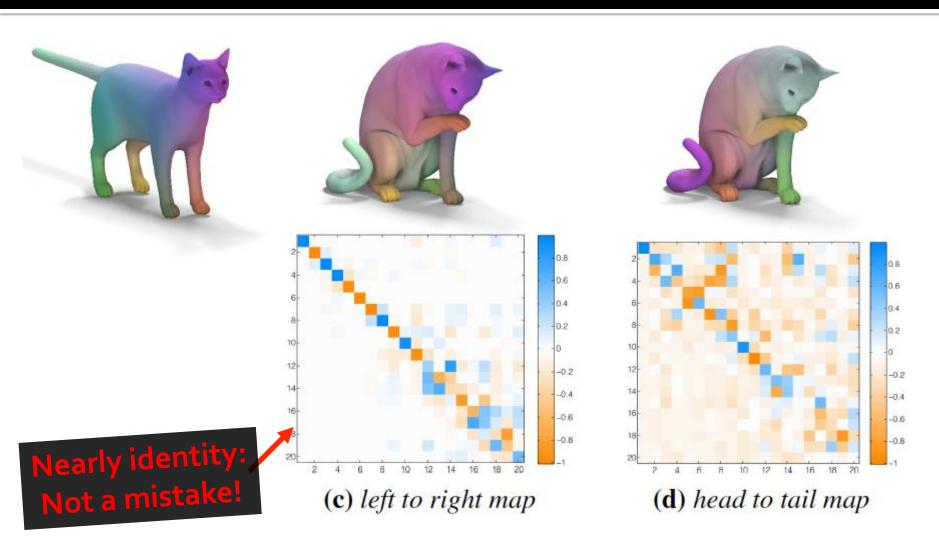
[Ovsjanikov et al. 2012]



Functional map:

Matrix taking Laplace-Beltrami (Fourier) coefficients on M to coefficients on  $M_{\circ}$ 

# Example Maps



#### **Functional Maps**

- Simple Algorithm
  - Compute some geometric functions to be preserved: A, B
  - Solve in least-squares sense for C: B = C A
- Additional Considerations
  - Favor commutativity
  - Favor orthonormality (if shapes are isometric)
  - Efficiently getting point-to-point correspondences

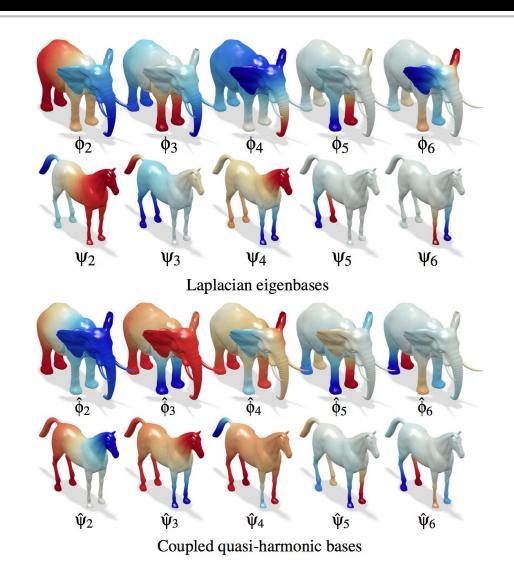
#### Tradeoff: Functional Maps

#### Pros:

- Condensed representation
- Linear
- Alternative perspective on mapping
- Many recent papers with variations
- Cons:
  - Hard to handle non-isometry Some progress in last few years!

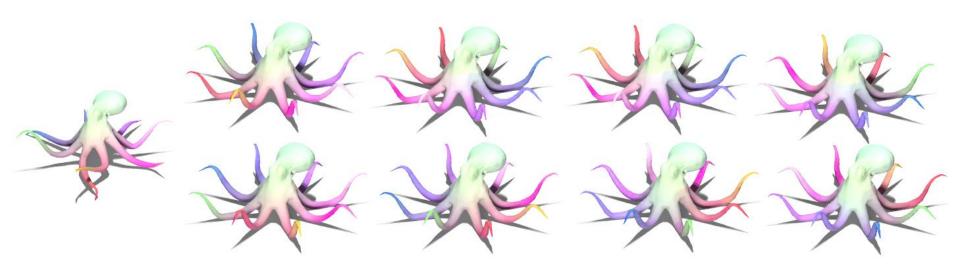
#### Example extension:

# Coupled Quasi-Harmonic Basis



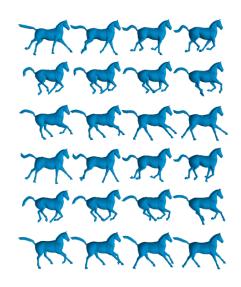
Example extension:

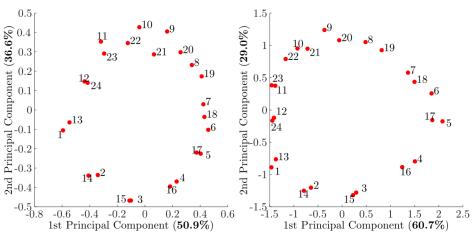
# Leverage Symmetry



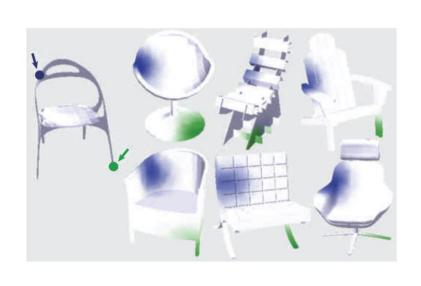
#### Example extension:

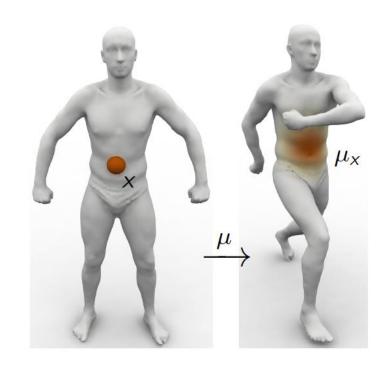
# **Analyze Deformation**

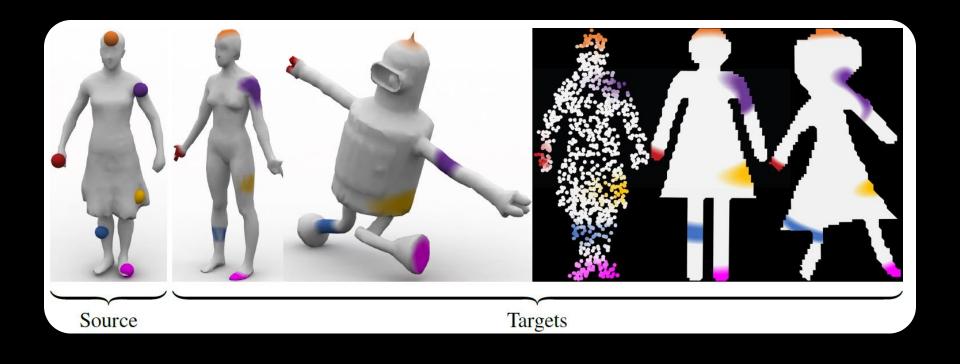




# Soft/Fuzzy Maps







# **Surface Correspondence**

Justin Solomon
MIT, Spring 2019

