Discrete Exterior Calculus

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Vector Calculus

\[ \text{div } \vec{v} = \nabla \cdot \vec{v} := \sum_i \frac{\partial v_i}{\partial x_i} \]

\[ \text{curl } \vec{v} = \nabla \times \vec{v} := \cdots \]

\[ \Delta f = \nabla \cdot \nabla f := \sum_i \frac{\partial^2 f}{\partial x_i^2} \]
Famous Theorems (in $\mathbb{R}^2$)

\[ \int_{\Omega} \text{div} \, \vec{v} \, dA = \int_{\partial \Omega} \vec{v} \cdot \vec{n} \, dl \]

"Divergence Theorem"

\[ \int_{\Omega} \text{curl} \, \vec{v} \, dA = \int_{\partial \Omega} \vec{v} \cdot \vec{t} \, dl \]

"Green’s Theorem"
Famous Theorems (in $\mathbb{R}^2$)

\[ \int_{\Omega} \text{div} \, \vec{v} \, dA = \int_{\partial\Omega} \vec{v} \cdot \vec{n} \, dl \]

“Divergence Theorem”

\[ \int_{\Omega} \text{curl} \, \vec{v} \, dA = \int_{\partial\Omega} \vec{v} \cdot \vec{t} \, dl \]

“Green’s Theorem”
Even Simpler Example...

\[ \int_{a}^{b} f'(x) \, dx = f(b) - f(a) \]
Pattern?

\[ \int_{\text{region}} \text{[derivative]} \, dV = \int_{\text{boundary}} \text{[quantity]} \, dA \]
Extension of vector calculus to surfaces (and manifolds).
1. **Exterior calculus**
   Alternating $k$-forms, derivatives, and integration

2. **Discrete exterior calculus**
   All that, on a simplicial complex
Our goal: Semester course in 2.5 lectures…
1. **Exterior calculus**
   Alternating $k$-forms, derivatives, and integration

2. **Discrete exterior calculus**
   All that, on a simplicial complex
New Rule

Everything must be intrinsic!

Vector fields are tangent!
Dual of a Vector Space $V$

\[ V^* := \{\text{linear maps } \xi : V \to \mathbb{R} \} \]

**Property:**

$V$, $V^*$ have same dimension.

\[ \{e_i\} \text{ basis for } V \implies \{dx^i\} \text{ basis for } V^* \]

\[ dx^i(e_j) := \begin{cases} 1 & \text{if } i = j \\ 0 & \text{otherwise} \end{cases} \]
One-Form: Dual of a Vector

\[ \omega(w) := v \cdot w \]
Intuition

Needle in a 1-form onion

https://www.aliexpress.com/price/needle-shredder_price.html

$$\omega(v) = \text{number of layers}$$
More Intuition

\[ \mathbf{v} \cdot \mathbf{w} = (v_1 \ v_2 \ \cdots \ v_n) \begin{pmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{pmatrix} \]

Row vs. column vectors
$g_{ij} := \langle e_i, e_j \rangle$, $g^{ij} := (g^{-1})_{ij}$

$v := \sum_i v^i e_i \quad w := \sum_j w^j e_j$

$\langle v, w \rangle = \sum_{ij} v^i w^j \langle e_i, e_j \rangle$

$:= \sum_{ij} v^i w^j g_{ij}$

$= v^i w^j g_{ij}$

Sum over repeated indices!
Musical Isomorphisms: Flat

\[ v^b (w) := \langle v, w \rangle \]

\[
\begin{align*}
    v &= \sum v^i e_i \\
    v^b &= \sum \omega_i dx^i \\
    \omega_i &= \sum_j g_{ij} v^j
\end{align*}
\]

Vector to covector (lowers index)
Musical Isomorphisms: Sharp

\[ \xi (w) := \langle \xi^\# , w \rangle \]

\[ \omega = \sum \omega_i dx^i \]

\[ \omega^\# = \sum v^i e_i \]

\[ v^i = \sum g^{ij} \omega_j \]

Covector to vector (raises index)
Forms on Surfaces

\[ f : \sum \rightarrow \mathbb{R} \]

o-form
Differential One-Forms

**Vector field**
\[ \vec{u} : \Sigma \rightarrow T\Sigma \]

**1-form**
\[ \omega(\vec{x}) = \vec{u} \cdot \vec{x} \]
Evaluating One-Forms

\[ \omega(\vec{v}) = \sum_{i} \omega^i v_i \]

No metric matrix \( g \)
Q: What is a two-form?
Continuing Onion Analogy

\[ \omega(v, w) = \text{number of oriented cells} \]
Interlude: Line integral

\[ W \approx \int_{\gamma} \vec{F} \cdot d\vec{l} \]

Misleading notation!

Work = force * distance

Interlude: Line integral

\[ W \approx \sum_i \omega x_i (\Delta x_i) \]

A hope of making rigorous: Resembles Riemann integral

Work = force * distance
Bilinear (same as 1D):

\[
\begin{align*}
\omega_x(c\Delta x_1, \Delta x_2) &= c\omega_x(\Delta x_1, \Delta x_2) \\
\omega_x((\Delta x_1 + \Delta x'_1), \Delta x_2) &= \omega_x(\Delta x_1, \Delta x_2) + \omega_x(\Delta x'_1, \Delta x_2) \\
\omega_x(\Delta x_1, c\Delta x_2) &= c\omega_x(\Delta x_1, \Delta x_2) \\
\omega_x(\Delta x_1, \Delta x_2 + \Delta x'_2) &= \omega_x(\Delta x_1, \Delta x_2) + \omega_x(\Delta x_1, \Delta x'_2)
\end{align*}
\]

Flux through degenerate window:

\[
\omega_x(\Delta x, \Delta x) = 0
\]

Notice: Signed!

\[
F \approx \sum_i \omega_{x_i}(\Delta x_{i1}, \Delta x_{i2})
\]
Bilinear (same as 1D):

\[ \omega_x(c \Delta x_1, \Delta x_2) = c \omega_x(\Delta x_1, \Delta x_2) \]

\[ \omega_x((\Delta x_1 + \Delta x_1'), \Delta x_2) = \omega_x(\Delta x_1, \Delta x_2) + \omega_x(\Delta x_1', \Delta x_2) \]

\[ \omega_x(\Delta x_1, c \Delta x_2) = c \omega_x(\Delta x_1, \Delta x_2) \]

\[ \omega_x(\Delta x_1, \Delta x_2 + \Delta x_1') = \omega_x(\Delta x_1, \Delta x_2) + \omega_x(\Delta x_1, \Delta x_1') \]

Flux through degenerate window:

\[ \omega_x(\Delta x, \Delta x) = 0 \]

Anti-symmetric (follows from properties above):

\[ \omega_x(\Delta x_1, \Delta x_2) = -\omega_x(\Delta x_2, \Delta x_1) \]
Bilinear:

\[
\omega_x(c\Delta x_1, \Delta x_2) = c\omega_x(\Delta x_1, \Delta x_2)
\]
\[
\omega_x((\Delta x_1 + \Delta x_1'), \Delta x_2) = \omega_x(\Delta x_1, \Delta x_2) + \omega_x(\Delta x_1', \Delta x_2)
\]
\[
\omega_x(\Delta x_1, c\Delta x_2) = c\omega_x(\Delta x_1, \Delta x_2)
\]
\[
\omega_x(\Delta x_1, \Delta x_2 + \Delta x_2') = \omega_x(\Delta x_1, \Delta x_2) + \omega_x(\Delta x_1, \Delta x_2')
\]

Flux through degenerate window:

\[
\omega_x(\Delta x, \Delta x) = 0
\]

Alternative equivalent definition:

\[
\omega_x(\Delta x_1, \Delta x_2) = -\omega_x(\Delta x_2, \Delta x_1)\text{(alternating)}
\]

**k-form:** Same thing, *k* slots!
More Concrete 2-Forms on $\mathbb{R}^n$

$\omega \in \Lambda^2 \implies \omega(v, w) = v^\top M w$

where $M^\top = -M$ (“antisymmetric”)

**In 2D:**

$\omega(v, w) = c \cdot v^\top \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} w$

- One DOF
- 90° rotation

**In 3D:**

$\omega(v, w) = v^\top \begin{pmatrix} 0 & -c_3 & c_2 \\ c_3 & 0 & -c_1 \\ -c_2 & c_1 & 0 \end{pmatrix} w = -c \cdot (v \times w)$

- Three DOFs
- DOFs agree with cross product!
For each point $p$ on a surface:

$k$ vectors in the tangent space at $p$ \(\rightarrow\) \textit{Differential $k$-form} \(\rightarrow\) $k$-linear Alternating
Two relevant details:
- $k =$ number of inputs
- $n =$ dimension

e.g. "a 2-form over $\mathbb{R}^3$"

($k=2, n=3$)
Alternating $k$-Forms as Flux Sensors

One-form: \[ \omega(\Delta x) = \text{how much flux in direction } \Delta x \]

Two-form: \[ \omega(\Delta x_1, \Delta x_2) = \text{how much flux in parallelogram } (\Delta x_1, \Delta x_2) \]

Some Algebra

On the board:
Space of $k$-forms on $\mathbb{R}^k$ is one-dimensional.

On the board:
$k$-forms on $\mathbb{R}^n$ are zero when $k > n$.

Area form: $dA$
Cross product of vectors is weird!

**Products: Observations About \( \times \)**

The units change:

\[ \text{inches} \times \text{inches} = \text{inches}^2 \]

Not product-like behavior:

\[ \vec{x} \times \vec{x} = \vec{0} \]

Dimensionality of cross product is variable:

\[ 2\vec{D} \times 2\vec{D} = \text{scalar} \]
\[ 3\vec{D} \times 3\vec{D} = \text{vector} \]

Cross product of vectors is weird!
Wedge: Product of Onions
Wedge: Product of Onions

\[ \varepsilon \wedge \eta \wedge \omega \]
Wedge Product of One-Forms

Key idea: 2-form measures size of parallelogram projected onto some (oriented) plane
Wedge of One-Forms

\[ \alpha \wedge \beta(u, v) = \alpha(u)\beta(v) - \alpha(v)\beta(u) \]

Build 2D flux sensor out of 1D flux sensors

\[ \alpha \wedge \beta = -\beta \wedge \alpha \]
$\alpha(w) := a \cdot w$

$\beta(w) := b \cdot w$

$\implies (\alpha \wedge \beta)(u, v) = \alpha(u)\beta(v) - \alpha(v)\beta(u)$

$= (a \cdot u)(b \cdot v) - (a \cdot v)(b \cdot u)$

$= (a \times b) \cdot (u \times v)$

*For one-forms:*

“How similar is parallelogram (a,b) to parallelogram (u,v)?”

*Notice: All 2-forms are wedges of 1-forms.*
Symbol of a Permutation

\[ \varepsilon(P) = \begin{cases} 
+1 & \text{if } P \text{ has an even number of swaps} \\
-1 & \text{otherwise}
\end{cases} \]

Examples:

(1234)
(1324)
(1342)
Wedge Product: Formal Definition

\[ \alpha \in \Lambda^k, \beta \in \Lambda^\ell \]

\[ \alpha \wedge \beta(v_1, \ldots, v_{k+\ell}) = \frac{1}{k!\ell!} \sum_{\sigma \in \text{Perm}(k+\ell)} \varepsilon(\sigma) \alpha(v_1, \ldots, v_k) \beta(v_{k+1}, \ldots, v_{k+\ell}) \]

\[ \in \Lambda^{k+\ell} \]

Antisymmetry: \[ \alpha \wedge \beta = (-1)^{k\ell} \beta \wedge \alpha \]

Associativity: \[ \alpha \wedge (\beta \wedge \gamma) = (\alpha \wedge \beta) \wedge \gamma \]

Distributivity: \[ \alpha \wedge (\beta + \gamma) = \alpha \wedge \beta + \alpha \wedge \gamma \]

\[ \implies \alpha \wedge \alpha \equiv 0 \]
Basis for $k$-Forms

$$dx_{i_1} \wedge dx_{i_2} \wedge \cdots \wedge dx_{i_k}$$

with no repeated indices.
Inner Product of 1-Forms

\[ \langle \xi, \eta \rangle := \langle \xi^\#, \eta^\# \rangle \]
Inner Product of $k$-Forms

$\langle \alpha_1 \land \cdots \land \alpha_k, \beta_1 \land \cdots \land \beta_k \rangle := \det(\langle \alpha_i, \beta_j \rangle)$

Example: Inner product of 2-forms over $\mathbb{R}^3$

$\langle v^b \land w^b, a^b \land b^b \rangle = \det \begin{pmatrix} v \cdot a & v \cdot b \\ w \cdot a & w \cdot b \end{pmatrix}$

$= (v \times w) \cdot (a \times b)$

$[= v^b \land w^b (a, b)]$

Again!

“How similar is parallelogram $(v, w)$ to parallelogram $(a, b)$?”
\( \alpha \wedge (\star \beta) = \langle \alpha, \beta \rangle dA \)

**Hodge Star (\(\star\))**

\(\star(u \wedge v) = w\)

**Analogy:** orthogonal complement
**Key differences:** orientation & “finite extent”
**Small detail:** \(z \wedge \star z\) is positively oriented

\(k \mapsto (n - k)\)

Image courtesy K. Crane
Hodge Star in 2D

Prove on board

Image courtesy K. Crane
Differential $k$-Forms on Manifolds

\[ \Lambda^k := \{ \text{alternating } k\text{-multilinear forms} \} \]

\[ \Omega^k := \{ \omega \text{ taking } p \in \Sigma \mapsto \Lambda^k(T_p \Sigma) \} \]

“One differential form per tangent plane”
Inner Product of $k$-Forms

\[ \langle \alpha, \beta \rangle := \int_{\Sigma} \star \alpha \wedge \beta \]
Definition (Differential). Suppose \( \varphi : M \rightarrow N \) is a map from a submanifold \( M \subseteq \mathbb{R}^k \) into a submanifold \( N \subseteq \mathbb{R}^\ell \). Then, the differential \( d\varphi_p : T_pM \rightarrow T_{\varphi(p)}N \) of \( \varphi \) at a point \( p \in M \) is given by

\[
d\varphi_p(v) := (\varphi \circ \gamma)'(0),
\]

where \( \gamma : (-\epsilon, \epsilon) \rightarrow M \) is any curve with \( \gamma(0) = p \) and \( \gamma'(0) = v \in T_pM \).

Linear map of tangent spaces

\[
d\varphi_p (\gamma'(0)) := (\varphi \circ \gamma)'(0)
\]
**Definition (Differential).** Suppose \( \varphi : M \to N \) is a map from a submanifold \( M \subseteq \mathbb{R}^k \) into a submanifold \( N \subseteq \mathbb{R}^\ell \). Then, the differential \( d\varphi_p : T_p M \to T_{\varphi(p)} N \) of \( \varphi \) at a point \( p \in M \) is given by

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**Linear map of tangent spaces**

\[
d\varphi_p \left( \gamma'(0) \right) := (\varphi \circ \gamma)'(0)
\]

"Exterior derivative of o-form"
Fancy Notation

\[ \nabla f = (df)^\# \]

\[ (\nabla f)^b = df \]
Construction of Exterior Derivative

Given a 1-form $\alpha$, when is there a function $f$ with $\alpha = df$?

\[
\alpha := \sum_i f_i dx^i
\]

\[
\frac{\partial^2 f}{\partial x^i \partial x^j} = \frac{\partial^2 f}{\partial x^j \partial x^i} \quad \Rightarrow \quad \frac{\partial f_j}{\partial x^i} = \frac{\partial f_i}{\partial x^j}
\]

\[
\Rightarrow 0 = \sum_{ij} \frac{\partial f_j}{\partial x^i} dx^i \wedge dx^j
\]

Transforms $d$ on 0-forms to $d$ on 1-forms...

Iterate!

Alternating!
Exterior Derivative: Axiomatic

Differential: $df$ agrees with directional derivative

Product rule: $d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^k \alpha \wedge d\beta$

Exactness: $d^2 = 0$
Product Rule: Intuition

\[(fg)' = f'g + fg'\]

Images courtesy K. Crane
Integration of $k$-Forms

Integrate on $k$-dimensional objects

\[ \int_{\gamma} \omega := \int_{\gamma} \omega(T) \, ds \]

Measures amount of $\omega$ parallel to $\gamma$
Stokes’ Theorem

\[ \int_{\Omega} d\omega = \int_{\partial \Omega} \omega \]
Intuition for Exactness

\[ d^2 = 0 \]

\[
\int_{\Omega} d^2 \omega = \int_{\partial \Omega} d\omega \\
= \int_{\partial \partial \Omega} \omega \\
= \int_{\{\}} \omega = 0
\]
\[ \nabla f = (df)^\# \]

\[ \nabla \cdot F = \star d \star (F^\dagger) \]

\[ \nabla \times F = (\star d(F^\dagger))^\# \]

\[ \Delta f = \star d \star df \]

Homework!
1. **Exterior calculus**  
   Alternating $k$-forms, derivatives, and integration

2. **Discrete exterior calculus**  
   All that, on a simplicial complex
Discrete version of exterior calculus.

\[ \omega^b \wedge \omega_1 \wedge \omega_2 \ast \omega \, d\omega \]
Recall: Oriented Simplicial Complex
Recall:

Dual Complex
The Trick

Store *integrals of forms*!
Integrated $k$-forms

Discrete $0$-form

$$\int_v \omega = f(v) \in \mathbb{R}^{\mid V \mid}$$

Store integrated quantities!
Integrated $k$-forms

Discrete 1-form

\[ \int_e \omega \in \mathbb{R}^{|E|} \]

Store integrated quantities!
Integrated $k$-forms

Discrete 2-form

\[ \int_t \omega \in \mathbb{R}^{\mid T \mid} \]
Exterior Derivative

\[ \int_\Omega d\omega = \int_{\partial \Omega} \omega \]

Stokes’ Theorem

\[ \int_e d\omega = \int_{\partial e} \omega = \omega_2 - \omega_1 \]

\[ d_{01} \in \{-1, 0, 1\} \quad \left| \begin{array}{c} E \end{array} \right| \times \left| \begin{array}{c} V \end{array} \right| \]
Exterior Derivative

\[ d \in \mathbb{R}^{|E| \times |V|} \]

consists of 1, 0, -1

\[
\int_e d\omega = \int_{\partial e} \omega = \omega_2 - \omega_1
\]
Exterior Derivative

\[ d \in \mathbb{R}^{\{|F| \times |E|\}} \]

consists of 1, 0, -1

\[
\int_t d\omega = \int_{\partial t} \omega = \omega_1 - \omega_2 + \omega_3
\]
Exterior Derivative

\[ d \in \mathbb{R}^{F \times |E|} \]

Haven’t made any approximations yet!

\[ \int_{t} d\omega = \int_{\partial t} \omega = \omega_{1} - \omega_{2} + \omega_{3} \]
Observation

“$d^2 = 0$”

Two different $d$ matrices

You proved this in your homework!
Hodge Star: Idea

Moves to dual mesh
Moves to dual mesh

Primal 2-form

Dual 0-form

Hodge Star
Hodge Star

Primal 1-form

Dual 1-form

Moves to dual mesh
Hodge Star Matrices

primal

dual

Image courtesy K. Crane
Hodge Star Matrices

This is where approximations appear.
Primal 2-Form / Dual 0-Form

\[ \star_{ii} = \text{Area}(\text{triangle } i)^{-1} \]
Primal 1-Form / Dual 1-Form

What do you think it is?

Careful with orientation/sign!

\[ \star \omega = \frac{|e_\star|}{|e|} \omega \]

Ratio of edge lengths
Primal 1-Form / Dual 1-Form

\[ \frac{|e^*_{ij}|}{|e_{ij}|} = \frac{1}{2} (\cot \alpha_j + \cot \beta_j) \]

Choice of dual: Circumcenter
Weighted Triangulations for Geometry Processing

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In this paper, we investigate the use of weighted triangulations as discrete, augmented approximations of surfaces for digital geometry processing. By incorporating a scalar weight per mesh vertex, we introduce a new notion of discrete metric that defines an orthogonal dual structure for arbitrary triangle meshes and thus extends weighted Delaunay triangulations to surface meshes. We also present alternative characterizations of this primal-dual structure (through combinations of angles, areas, and lengths) and, in the process, uncover closed-form expressions of mesh energies that were previously known in implicit form only. Finally, we demonstrate how weighted triangulations provide a faster and more robust approach to a series of geometry processing applications, including the generation of well-centered meshes, self-supporting surfaces, and sphere packing.

Categories and Subject Descriptors: I.3.5 [Computer Graphics]: Computational Geometry and Object Modeling—Curve, surface, solid, and object representations

Additional Key Words and Phrases: discrete differential geometry, discrete metric, weighted triangulations, orthogonal dual diagram.

1. INTRODUCTION

Triangle meshes are arguably the predominant discretization of surfaces in graphics, and by now there is a large body of literature on the theory and practice of simplicial meshes for computations. However, many geometry processing applications rely, overtly or covertly, on an orthogonal dual structure to the primal mesh. The use of such a dual structure is very application-dependent, with circumcentric and power duals being found, for instance, in physical simulation [Elcott et al. 2007; Batty et al. 2010], architecture modeling [Liu et al. 2013; de Goes et al. 2013] and parameterization [Mercat 2001; Jin et al. 2008]. While most of these results are limited to planar triangle meshes, little attention has been paid to exploring orthogonal duals for triangulated surface meshes.

In this paper, we advocate the use of orthogonal dual structures to enrich simplicial approximations of arbitrary surfaces. We introduce an extended definition of metric for these discrete surfaces with which one can not only measure length and area of simplices, but also the orientation and the curvature of the surface they encode.
Primal o-Form / Dual 2-Form

\[ \star ii = \text{Area}(\text{triangle } i) \]
Recall:
Barycentric Lumped Mass

Area/3 to each vertex

http://www.alecjacobson.com/weblog/?p=1146
Additional Options

Barycentric cell

$C_i = \text{barycenter of triangle}$

Voronoi cell

$C_i = \text{circumcenter of triangle}$

Mixed cell
If $\theta < \pi/2$, $c_i$ is the circumcenter of the triangle $(v_i, v, v_{i+1})$.

If $\theta \geq \pi/2$, $c_i$ is the midpoint of the edge $(v_i, v_{i+1})$.

$$A(v) = \sum_{v_i \in N(v)} \left( \text{Area}(c_i, v, (v + v_i)/2) + \text{Area}(c_{i+1}, v, (v + v_i)/2) \right)$$
HOT: Hodge-Optimized Triangulations

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Abstract

We introduce Hodge-optimized triangulations (HOT), a family of well-shaped primal-dual pairs of complexes designed for fast and accurate computations in computer graphics. Previous work most commonly employs barycentric or circumcentric duals; while barycentric duals guarantee that the dual of each simplex lies within the simplex, circumcentric duals are often preferred due to the induced orthogonality between primal and dual complexes. We instead promote the use of weighted duals ("power diagrams"). They allow greater flexibility in the location of dual vertices while keeping primal-dual orthogonality, thus providing a valuable extension to the usual choices of dual by only adding one additional scalar per primal vertex. Furthermore, we introduce a family of functionals on pairs of complexes that we derive from bounds on the errors induced by diagonal Hodge stars, commonly used in discrete computations. The minimizers of these functionals, called HOT meshes, are shown to be generalizations of Centroidal Voronoi Tessellations and Optimal Delaunay Triangulations, and to provide increased accuracy and flexibility for a variety of computational purposes.

Keywords: Optimal triangulations, Discrete Exterior Calculus, Discrete Hodge Star, Optimal Transport.

Links: 🌐DL  📄PDF  📲WEB

1 Introduction

Figure 1: Primal/Dual Triangulations: Using the barycentric dual (top-left) does not generally give dual meshes orthogonal to the primal mesh. Circumcentric duals, both in Centroidal Voronoi Tessellations (CVT, top-middle) and Optimal Delaunay Triangulations (ODT, top-right), can lead to dual points far from the barycenters of the triangles (blue points). Leveraging the freedom provided by weighted circumcenters, our Hodge-optimized triangulations (HOT) can optimize the dual mesh alone (bottom-left) or both the primal and dual meshes (bottom-right), e.g., to make them more orthogonal.
Discrete deRham Complex

0-forms (vertices) 1-forms (edges) 2-forms (faces) 3-forms (tets)

\[ \delta \quad \text{d} \quad \delta \quad \text{d} \quad \delta \quad \text{d} \]

\[ \star \quad \star^{-1} \quad \star \quad \star^{-1} \quad \star \quad \star^{-1} \]

\[ \text{d} \quad \text{d} \quad \text{d} \quad \text{d} \]

In Practice

- Build up tons of matrices
- Multiply them together for complicated operators

\[ d_{01}, d_{12}, \star 02, \cdots \]
Dot product:
One primal, one dual.
(Already integrated!)
Co-Differential

\[
\langle d\beta, \alpha \rangle = - \langle \beta, \star d \star \alpha \rangle
\]

\[\delta := - \star d \star\]
Yet Another Cotan Laplacian

\[ L = d_{12} \star_{11} d_{01} \]

\[ M = \star_{02} \]
Hodge Laplacian

\[ \Delta := d \star d \star + \star d \star d \]

What happens for 0-forms? 2-forms on a surface?
Whitney Elements

\[ \phi_{ij}(p) = \phi_i(p) d\phi_j - \phi_j(p) d\phi_i, \quad (d\phi_i)^\# = \nabla \phi_i \]

Interpolate one-form over triangle

Discontinuous along edges!
Helmholtz-Hodge Decomposition

\[ \omega = \delta \beta + d \alpha + \gamma \]

where \( d \gamma = 0 \), \( \delta \gamma = 0 \)

Divergence free

Curl free

Harmonic
Computing the Decomposition

\[ \omega = \delta \beta + d\alpha + \gamma \]

where \( d\gamma = 0, \delta \gamma = 0 \)

\[ \delta d\alpha = \delta \omega \]

\[ d\delta \beta = d\omega \]

\[ \gamma = \omega - \delta \beta - d\alpha \]
Conclusion: For $\lambda \neq 0$, they’re obtained by $d$ and $\star$ of Laplacian eigenfunctions.
The Helmholtz-Hodge Decomposition - A Survey

Harsh Bhatia, Student Member IEEE, Gregory Norgard, Valerio Pascucci, Member IEEE, and Peer-Timo Bremer, Member IEEE

Abstract—The Helmholtz-Hodge Decomposition (HHD) describes the decomposition of a flow field into its divergence-free and curl-free components. Many researchers in various communities like weather modeling, oceanology, geophysics and computer graphics are interested in understanding the properties of flow representing physical phenomena such as incompressibility and vorticity. The HHD has proven to be an important tool in the analysis of fluids, making it one of the fundamental theorems in fluid dynamics. The recent advances in the area of flow analysis have led to the application of the HHD in a number of research communities such as flow visualization, topological analysis, imaging, and robotics. However, since the initial body of work, primarily in the physics communities, research on the topic has become fragmented with different communities working largely in isolation often repeating and sometimes contradicting each others results. Additionally, different nomenclature has evolved which further obscures the fundamental connections between fields making the transfer of knowledge difficult. This survey attempts to address these problems by collecting a comprehensive list of relevant references and examining them using a common terminology. A particular focus is the discussion of boundary conditions when computing the HHD. The goal is to promote further research in the field by creating a common repository of techniques to compute the HHD as well as a large collection of example applications in a broad range of areas.

Index Terms—Vector fields, Incompressibility, Boundary Conditions, Helmholtz-Hodge decomposition.
Recommended Reading

Today will take a few random samples
Fig. 2. Sequence of images from the Hurricane Luis sequence, with eye segmented

Fig. 1. (a) Motion field in a anticlockwise rotating hurricane sequence extracted using the BMA. (b) The divergence free potential function with a distinct maximum and corresponding contours.

Fluid Simulation

Stam. “Stable Fluids.” SIGGRAPH 1999. (and many others)

Incompressible: No divergence
Vector Field Editing

Tong et al. “Discrete Multiscale Vector Field Decomposition.”
TOG 2003.
Separate turbulence from acoustics in solar simulation

Reconstruct VF from Noisy Samples

\[ \Phi_{df}(x) = H\phi(x) - tr\{H\phi(x)\}I \]
\[ \Phi_{cf}(x) = -H\phi(x) \]

Macedo and Castro.
Extension to Smooth Surfaces

Subdivision Exterior Calculus for Geometry Processing

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Pixar Animation Studios

Mathieu Desbrun
Caltech

Mark Meyer
Pixar Animation Studios

Tony DeRose
Pixar Animation Studios

Figure 1: Subdivision Exterior Calculus (SEC). We introduce a new technique to perform geometry processing applications on subdivision surfaces by extending Discrete Exterior Calculus (DEC) from the polygonal to the subdivision setting. With the preassembly of a few operators on the control mesh, SEC outperforms DEC in terms of numerics with only minor computational overhead. For instance, while the spectral conformal parameterization [Mullen et al. 2008] of the control mesh of the mannequin head (left) results in large quasi-conformal distortion (mean = 1.784, max = 9.4) after subdivision (middle), simply substituting our SEC operators for the original DEC operators significantly reduces distortion (mean = 1.005, max = 3.0) (right). Parameterizations, shown at level 1 for clarity, exhibit substantial differences.

Abstract

This paper introduces a new computational method to solve differential equations on subdivision surfaces. Our approach adapts the numerical framework of Discrete Exterior Calculus (DEC) from the polygonal to the subdivision setting by exploiting the refractivity of subdivision basis functions. The resulting Subdivision Exterior Calculus (SEC) provides significant improvements in accuracy compared to existing polygonal techniques, while offering exact finite-dimensional analogs of continuum structural identities such as Stokes' theorem and Helmholtz-Hodge decomposition. We demonstrate the versatility and efficiency of SEC on common geometry processing tasks including parameterization, geodesic distance computation, and vector field design.

Keywords: Subdivision surfaces, discrete exterior calculus, differential geometry processing.
What About Symmetric Tensors?

Discrete 2-Tensor Fields on Triangulations

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Abstract

Geometry processing has made ample use of discrete representations of tangent vector fields and antisymmetric tensors (i.e., forms) on triangulations. Symmetric 2-tensors, while crucial in the definition of inner products and elliptic operators, have received only limited attention. They are often discretized by first defining a coordinate system per vertex, edge or face, then storing their components in this frame field. In this paper, we introduce a representation of arbitrary 2-tensor fields on triangle meshes. We leverage a coordinate-free decomposition of continuous 2-tensors in the plane to construct a finite-dimensional encoding of tensor fields through scalar values on oriented simplices of a manifold triangulation. We also provide closed-form expressions of pairing, inner product, and trace for this discrete representation of tensor fields, and formulate a discrete covariant derivative and a discrete Lie bracket. Our approach extends discrete finite-element exterior calculus, recovers familiar operators such as the weighted Laplacian operator, and defines discrete notions of divergence-free, curl-free, and traceless tensors—thus offering a numerical framework for discrete tensor calculus on triangulations. We finally demonstrate the robustness and accuracy of our operators on analytical examples, before applying them to the computation of anisotropic geodesic distances on discrete surfaces.

Categories and Subject Descriptors (according to ACM CCS): Computer Graphics [1.3.5]: Computational Geometry and Object Modeling—Curve and surface representations.
## Summary

<table>
<thead>
<tr>
<th>Pros</th>
<th>Cons</th>
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<tbody>
<tr>
<td>• Coordinate-free representation using only one scalar value per edge.</td>
<td>• Discontinuous reconstruction for low-order Whitney basis functions.</td>
</tr>
<tr>
<td>• Simple interpolation of edge values.</td>
<td>• No clear vector at vertices, so incompatible with vertex-based deformation of meshes.</td>
</tr>
<tr>
<td>• Simple differential operators leveraging the DEC literature.</td>
<td>• Generalization to $n$-vector fields has not been studied.</td>
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</tbody>
</table>

From *Vector Field Processing on Triangle Meshes*  
de Goes, Desbrun, and Tong (SIGAsia 2015)
Discrete Exterior Calculus

Justin Solomon
MIT, Spring 2019

Original version from Stanford CS 468, spring 2013 (Butscher & Solomon)