Sample-Based HZD Control for Robustness and Slope Invariance of Planar Passive Bipedal Gaits

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Abstract—Controllers are developed that extend the hybrid zero dynamics (HZD) control approach to the control of planar biped walking by allowing constraints to be designed by sampling a gait of interest, rather than by optimization. The technique is used to enlarge the basin of attraction of a passive biped’s gait under the assumption that ideal actuation has been introduced at each of the body coordinates. In this case, no control effort is used at steady-state. The technique also enables the systematic modification of the gaits’ kinematic and dynamic properties. The main results are illustrated via two examples.

I. INTRODUCTION

A passive biped walker is a two-legged mechanism that is able to walk stably down a slope without active feedback control or energy input aside from gravity. Since McGeer first simulated and built such a mechanism in the 1980’s [1], passive biped walkers have had continued interest, primarily as a point of departure for building energetically efficient, actuated biped robots. Passive bipedal walkers, however, have two fundamentally limiting features. The first is that the basins of attraction associated with their orbits are small—meaning they are easily toppled. The second is a lack of variety of available walking motions; a gait’s features can only be changed by robot redesign or by ground slope change.

Actuation can remedy both of these shortcomings. Ideal actuation under active feedback control can be used to increase robustness and to change a gait’s characteristics, such as the minimum slope on which the biped is able to walk. Since the energetic cost of passive dynamic walking is, in fact, nonzero, the loss of stable passive gaits does not preclude the use of energetic efficiency as a metric in achieving a given objective, such as walking at a certain rate, walking on flat ground, or walking with increased robustness.

1Note that the addition of non-ideal actuation often results in the loss of all stable, passive gaits. This is because the usual means of actuating a biped is with actuators that are collocated with the biped’s joints. In such a configuration, the actuator’s dynamics are coupled with the biped’s. (An example where this does not occur is Collins’s powered 3D biped [2], which is powered by impulsive foot action.)

2Although there do theoretically exist stable gaits for passive bipeds at arbitrarily small slopes [3], the basins of attractions for gaits are impractically small.

3In walking down a slope, potential energy is consumed—potential energy that is the result of doing work to lift the mechanism to the top of the slope.

4In the HZD framework, the biped is assumed to have point contact with the ground and is therefore underactuated. With this assumption, the effective actuation that exists between the biped and the ground—because of unilateral constraints due to finite foot size—is made explicit. If a biped in question is, in fact, fully actuated, the HZD framework still applies. First an HZD controller is designed, and then an outer-loop control is designed that makes use of the ankle torque [8].

5When using non-ideal actuators, zero control effort is achieved in the sense that actuators perform no mechanical work on the system. With electrical motors, for example, electrical energy will be consumed to prevent friction and rotor inertia from doing work on the system.

Fig. 1. An underactuated two-link planar walking down a slope of incline \( \alpha \). The dynamics during the single support phase is that of the Acrobat [4].
feet, for presentation simplicity, the examples use the two-link walker depicted in Fig. 1, whose dynamics during the single support phase is that of the Acrobat [4].

A. Background and Motivation

Previous work on HZD control of planar robots with one degree of underactuation has produced, both in theory and experiment, walking gaits that have notably strong robustness properties. In HZD control, holonomic constraints are imposed on the actuated joints of the model [10], [11], resulting in a closed-loop system with dynamics and stability properties known a priori.

In previous work, the holonomic constraints (also called virtual constraints) have been chosen via numerical optimization over a pre-chosen, finitely parameterized set of constraints. This technique is acceptable when the objective of controller design is to induce a gait with certain stability and energetic properties. However, when the goal is to exactly achieve a certain steady-state motion—as will be the case here with the passive gaits—it is unlikely that a pre-chosen family of finitely parameterized holonomic constraints will be capable of reproducing the chosen motion.

This paper discusses an alternate method of designing HZD controllers—one without the use of a pre-chosen family of holonomic constraints. In essence, a given gait is sampled to obtain full-state information at chosen instants of time. Certain normalized quantities are computed from this full-state information and are used to define the holonomic constraints of an HZD controller. Splines are used to interpolate the normalized quantities between sample points. The nominal motion must be period-one and can be obtained from, for example, a passive gait or a gait induced by another, potentially unknown, control strategy.

Although the basin of attraction of the biped with the sample-based HZD controller may be larger, the closed-loop system will, in general, not be capable of achieving a variety of different gaits. To address this shortcoming, the notion of a constraint augmentation function is introduced. A constraint augmentation function is a finitely parameterized function, such as a polynomial, that gives a means to systematically modify a set of sample-based virtual constraints. As in previous work, these parameters may be chosen via optimization. Such augmentation functions may be used make gaits zero-slope capable or to modify any other kinematic or dynamic properties of the induced motion, while retaining, as much as possible, the robot’s original unactuated dynamic behavior.

B. Paper Outline

The content of the remainder of the paper is as follows. Sec. II presents the concepts of virtual constrains and HZD control in the context of walking on a sloped surface. Sec. III presents the developments of sample-based virtual constraints and their accompanying augmentation functions. Inline with the developments are two illustrative examples. Conclusions are drawn in Sec. IV. Animations of the examples are available on the web.

II. WALKING ON SLOPED GROUND

A. Model of Walking

The biped is assumed to be comprised of n rigid links connected by revolute joints such that (i) there are no closed kinematic chains; (ii) there are two symmetric legs and, possibly, a torso; and (iii) the leg ends contact the ground at a single point. The robot is said to be in single-support (or in the swing phase) when exactly one leg is in contact with the ground. The leg contacting the ground is called the stance leg and the other is called the swing leg. It is assumed that all of the biped’s internal degrees of freedom (DOFs) are actuated, but that the degree of freedom associated with the robot’s absolute orientation is left unactuated (i.e., no torque may be supplied between the robot and the ground). The swing-phase model is therefore underactuated.

The generalized coordinates of the biped are \( q := (q_a, q_u) \in Q \), where \( Q \) is an appropriate subset of \( \mathbb{R}^n \), \( q_u \) is the ground slope; see Fig. 1. The state of the biped be \( x := (q, \dot{q}) \in TQ \). Then, (1) may be written as

\[
\dot{x} = f(\alpha)(x) + g(x)u. \tag{2}
\]

The walking gait is assumed to be symmetric with respect to the two legs so that, in particular, the same swing phase model may be used irrespective of which leg is the stance leg.

Stance phases are separated by instantaneous phases of double support, occurring when both feet are in contact with the ground. This transition is modeled as an instantaneous, rigid-body collision [12], that occurs when \( x \in S := \{ x \in TQ \mid p_2(x) = 0 \} \). The transition model, which includes a permutation of the coordinates to account for the swapping of the legs’ roles, is algebraic and may be written as

\[
x^+ = \Delta(x^-), \tag{3}
\]

where the superscript “+” (resp. “−”) refers the value at the beginning (resp. end) of a step.

Throughout this paper, dependence on the ground-slope parameter is emphasized by the use of square brackets.
The overall model may be expressed as a single-charted hybrid model:

\[ \dot{x} = f(x) + g(x)u, \quad x^- \notin S \]
\[ x^+ = \Delta(x^-), \quad x^- \in S. \]  

(4)

Walking gaits will be analyzed as periodic orbits of the above model, with stability of a walking gait referring to stability of the corresponding periodic orbit. For formal definitions of solutions, orbits, and stability relating to (4), refer to [13].

B. Defining Virtual Constraints

Virtual constraints are defined as holonomic constraints that are imposed on the robot via feedback. These constraints are parameterized by a scalar function of the robot’s configuration, and, when enforced, they effectively reduce the closed-loop DOF of the robot. When virtual constraints satisfying certain invariance properties are exactly enforced the HZD of walking results.

To formally define virtual constraints, consider the following output on (2),

\[ \theta(q) : \mathcal{Q} \rightarrow R_\theta \subset \mathbb{R} \]  
\[ s(\theta) : R_\theta \rightarrow [0, 1] \]  
\[ h_d(s) : [0, 1] \rightarrow \mathbb{R}^{n-1} \]  
\[ y = h(q) := q_a - h_d \circ s \circ \theta(q) \]  

where \( \theta(q) \) is a function that is monotonic over a step and has a compact image \( R_\theta \), \( s(\theta) \) is a bijection with respect to \( R_\theta \) and normalizes \( \theta \) to the unit interval, and \( h_d(s) \) is a twice continuously differentiable function that gives the actuated coordinates of the robot. For notational simplicity, define \( \tilde{h}_d(\theta) := h_d \circ s(\theta) \) so that (5d) may be written

\[ y = q_a - \tilde{h}_d \circ \theta(q). \]  

(6)

Let \( \theta^+ \) and \( \theta^- \) denote, respectively, the values of \( \theta(q) \) at the beginning and the end of a step. Then, a valid choice for \( s \) is \( s(\theta) := (\theta - \theta^+)/(\theta^- - \theta^+) \). This choice will be assumed for the remainder of the paper.

Virtual constraints are said to be satisfied or enforced when \( y = 0 \). The constraint surface \( Z \) is defined as the subset of \( T\mathcal{Q} \) where the virtual constraints are satisfied,

\[ Z := \{ x \in T\mathcal{Q} \mid h(x) = 0, L_I h(x) = 0 \}. \]  

(7)

When viewed within the context of the hybrid model (4), the virtual constraints are required to have two types of invariance: inter-stride (or continuous-phase invariance) and intra-stride (or invariance across the impact event). Continuous-phase invariance refers to the property that once a solution of (4) is within the constraint surface, the solution remains in the constraint surface until the end of the single-support phase. This type of invariance is achieved by the appropriate design of an inter-stride feedback controller. The virtual constraints are invariant across the impact event if lying within the constraint surface before impact guarantees that the solution will lie within the constraint surface after the impact. This type of invariance is a property of virtual constraints themselves and is independent of the inter-stride feedback controller.

C. A Feedback yielding Continuous-Phase Invariance

Assume an output of the form (5), which may or may not be intra-stride invariant. The controller given in this subsection will render it continuous-phase invariant.

The controller’s development begins by taking the first two derivatives of the output,

\[ \dot{y} = \dot{q}_a - \frac{\partial \tilde{h}_d(\theta)}{\partial \theta} \dot{\theta} \] \hspace{1cm} (8a)
\[ \ddot{y} = \Upsilon(\theta) \ddot{q} - \frac{\partial^2 \tilde{h}_d(\theta)}{\partial \theta^2} \ddot{\theta}^2 \] \hspace{1cm} (8b)

where

\[ \Upsilon(\theta) := \begin{bmatrix} I & 0 \end{bmatrix} - \frac{\partial \tilde{h}_d(\theta)}{\partial \theta} c. \] \hspace{1cm} (9)

With (1), (8b) may be expressed as

\[ \ddot{y} = -\Upsilon(\theta) D^{-1}(q) F[\alpha](q, \dot{q}) - \frac{\partial^2 \tilde{h}_d(\theta)}{\partial \theta^2} \ddot{\theta}^2 \]
\[ \hspace{1cm} + \Upsilon(\theta) D^{-1}(q) B u. \] \hspace{1cm} (10)

The term \( L_\alpha L_f h(q, \dot{q}) \) is known as the decoupling matrix from the input \( u \) to the output \( y \). With the application of the input-output linearizing pre-feedback

\[ u = (L_\alpha L_f h(q, \dot{q}))^{-1} (v - L_\alpha^2 h[\alpha](q, \dot{q})), \] \hspace{1cm} (11)

the error dynamics (10) become \( \ddot{y} = v \). Thus, choosing \( v \) to be a PD controller,

\[ v = -K_p y - K_d \dot{y} \] \hspace{1cm} (12)

with poles sufficiently fast [14], the virtual constraints \( h_d(\theta) \) will be asymptotically enforced and continuous-phase invariant.

Remark 1: The control law, (11) and (12), requires measurement of \( (q, \dot{q}) \) and computation of \( L_\alpha L_f h(q, \dot{q}) \) and \( L_\alpha^2 h[\alpha](q, \dot{q}) \). While \( D(q), F[\alpha](q, \dot{q}) \), and \( B \) may be readily obtained from the system dynamics, the functions \( \tilde{h}_d(\theta), \partial \tilde{h}_d(\theta)/\partial \theta, \) and \( \partial^2 \tilde{h}_d(\theta)/\partial \theta^2 \) depend upon choice of virtual constraint.

D. The HZD of Walking

The HZD of walking is a subdynamic of the full hybrid walking model (4) that corresponds to the dynamics that are “left over” once the virtual constraints have been imposed. It is also a single-charted hybrid system, but of necessarily lower dimension than (4). The HZD resulting from virtual constraints based on (5) are developed next.

Let \( \sigma := D_n(q) \dot{q} \), where \( D_n \) is the last row of \( D \), and \( \theta := c \dot{q}, c \in \mathbb{R}^{1 \times n} \), where \( [I \ 0]' \), \( c \) is full rank. If \( y \equiv 0 \), then the robot’s configuration and velocity may be computed as

\[ q = \Phi_q(\theta), \quad \Phi_q(\theta) := \begin{bmatrix} I & 0 \end{bmatrix}^{-1} \begin{bmatrix} \tilde{h}_d(\theta) \\ 0 \end{bmatrix} \] \hspace{1cm} (13a)
\[ \dot{q} = \Phi_q(\theta) \sigma, \quad \Phi_q(\theta) := \begin{bmatrix} \Upsilon(\theta) \\ D_n(\theta) \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix}. \] \hspace{1cm} (13b)
With the output given by (5), and a few additional technical dynamics—the maximum dynamics that are compatible with $y \equiv 0$—are

$$
\begin{align*}
\dot{\theta} &= \kappa_1(\theta) \sigma, \quad \kappa_1(\theta) := e_\Phi(\theta), \\
\dot{\sigma} &= \kappa_2(\sigma, \theta), \quad \kappa_2(\sigma, \theta) := M_I g_0 x_{cm}(\alpha)(\theta),
\end{align*}
$$

(14a)

(14b)

where $M_I$ is the total mass of the biped, $g_0$ is the magnitude of the acceleration of gravity, and $x_{cm}$ is the horizontal position of the center of mass measured with respect to the stance leg end [15, Eq. (15)]. It may be shown that if the virtual constraints are intra-stride invariant, then at an impact,

$$
\sigma^+ = \delta_{zero} \sigma^-,
$$

(15)

where $\delta_{zero}$ is a constant readily computed using the definition of $\sigma$, (3), and (13). Taking $z := (\theta, \sigma)$ as a state vector, the single-charted HZD is,

$$
\Sigma_{zero} := \begin{cases} 
\dot{z} = f_{zero}(|\alpha|)(z), & z^- \notin S \cap Z \\
\dot{z}^+ = \Delta_{zero}(z^-), & z^- \in S \cap Z.
\end{cases}
$$

(16)

The HZD are said to be well-defined if the virtual constraints are invariant with respect impacts and solutions of the HZD are also solutions of the full system (4), i.e., the decoupling matrix, $L_g L_I h(q)$, is invertible along solutions of the HZD.

### F. Effects of Varying Ground Slope

The effects of varying the ground slope $\alpha$ on the existence of (stable) gaits are now presented. The presentation begins with two propositions summarizing several important facts.

**Proposition 1:** Under the assumption that the unactuated coordinate is measured relative to the walking surface, the following functions and surfaces are independent of $\alpha$:

- P1.1) the transition model, $\Delta(x)$,
- P1.2) the restricted switching surface, $S \cap Z$,
- P1.3) the restricted impact coefficient, $\delta_{zero}$, and
- P1.4) the decoupling matrix, $L_g L_I h(q)$.

**Proof:** Proof of P1.1 is trivial by inspection of [10, Eqns. 6 and 7]. P1.2 holds since $S$ is independent of $\alpha$, which is trivial by inspection, and because the output (5) is independent of $\alpha$. P1.3 holds by P1.1 and because $\sigma$ and (13) are independent of $\alpha$. P1.4 is trivial.

**Proposition 2:** Under the assumption that the unactuated coordinate is measured relative to the walking surface, if the HZD are well-defined for a given ground slope $\alpha$, then they will be defined for arbitrary $\alpha$.

**Proof:** Invariance of the virtual constraints with respect to $\alpha$ holds by P1.1 and P1.2. Invariance of the decoupling matrix with respect $\alpha$ holds by P1.4. Therefore, the proposition holds.

By Prop. 2, the minimum ground slope that a biped controlled by a sample-based HZD controller is capable of walking on can be determined by find the smallest $\alpha$ such that

$$
- \frac{V_{zero}(\alpha)(\theta^-)}{V_{max}(\alpha)} = \frac{1 - \delta_{zero}^2}{\delta_{zero}^2}.
$$

(23)

Note that the loss of stability amounts to the fixed point moving outside the restricted Poincaré map’s basin of attraction and, by P1.3, not a change in the map’s eigenvalue. Calculation of the maximum ground slope is more tedious and involves consideration of the ground reaction forces and actuator torque limits.

The next proposition gives an interesting observation regarding the loss of stability due to ground slope decrease.

**Proposition 3:** Equality of (23) is due to the change in the relative horizontal position of the COM, $x_{cm}$, over a step.

**Proof:** With (14b) the function $V_{zero}(\theta)$ may be written

$$
V_{zero}(\theta) = -2 \int_{\theta^+}^{\theta^-} M_I g_0 x_{cm}(\alpha)(\tau) \frac{c_\Phi(\tau)}{\delta_{zero}^2} d\tau.
$$

(24)

The function $\Phi_\theta(\tau)$ is independent of the absolute coordinate, and therefore independent of the ground slope, leaving $x_{cm}(\alpha)(\theta)$ as the only term dependent on $\alpha$.

### III. Development of New HZD-based Tools

#### A. Sample-Based Virtual Constraints

In a typical HZD controller design procedure, the output function, $h_d(\theta)$, is selected via numerical optimization from a
pre-chosen, finitely parameterized set of constraints. The first and second derivatives required by the controller, \( \partial \tilde{h}_d(\theta)/\partial \theta \) and \( \partial^2 \tilde{h}_d(\theta)/\partial \theta^2 \), are found in a straightforward manner by differentiating the output function itself.

Using the typical design method, controllers cannot be designed around a given, arbitrary gait since it is unlikely that the associated holonomic constraints will lie within the family used in optimization. In order to design an HZD controller around an arbitrary, period-one gait, an alternative method is required. One such method is described next.

**Proposition 4:** Assume an output of the form (5). Given a period-one periodic orbit of (4), the associated controller functions \( \tilde{h}_d(\theta) \), \( \partial \tilde{h}_d(\theta)/\partial \theta \), and \( \partial^2 \tilde{h}_d(\theta)/\partial \theta^2 \) can be approximated arbitrarily accurately, using interpolation techniques, without appealing to function differentiation or finite-difference methods.

**Proof:** Define \( \mathbf{q}(t) \) as the time evolution of the coordinates \( \mathbf{q} \) on the limit cycle. Similarly, define \( \dot{\mathbf{q}}(t), \ddot{\mathbf{q}}(t), \Theta(t), \dot{\Theta}(t), \ddot{\Theta}(t) \) as the time evolution of \( \dot{\mathbf{q}}, \ddot{\mathbf{q}}, \Theta, \dot{\Theta}, \ddot{\Theta} \) on the limit cycle. By monotonicity, \( \theta = \Theta(t) \) has a well-defined inverse, \( t = \Theta^{-1}(\theta) \).

Since the functions \( \dot{\mathbf{q}}(t), \ddot{\mathbf{q}}(t), \Theta(t), \dot{\Theta}(t), \ddot{\Theta}(t) \), and \( t = \Theta^{-1}(\theta) \) can each be approximated arbitrarily accurately using interpolation techniques (such as cubic splines) the following holds. On the periodic orbit, \( y \equiv 0 \), and thus (6) and (8) give

\[
\begin{align*}
\tilde{h}_d(\theta) &= \mathbf{q}_a(t)|_{t=\Theta^{-1}(\theta)} \\
\frac{\partial \tilde{h}_d}{\partial \theta}(\theta) &= \frac{\mathbf{q}_a(t)}{\Theta(t)}|_{t=\Theta^{-1}(\theta)} \\
\frac{\partial^2 \tilde{h}_d}{\partial \theta^2}(\theta) &= \left( \frac{\mathbf{q}_a(t)}{\Theta(t)} - \frac{\mathbf{q}_a(t) \dot{\Theta}(t)}{\Theta^2(t)} - \frac{\mathbf{q}_a(t) \ddot{\Theta}(t)}{\Theta^3(t)} \right)|_{t=\Theta^{-1}(\theta)}. \tag{25c}
\end{align*}
\]

Thus, given an existing limit cycle \( \mathbf{q}(t) \) for which a set of output functions \( h(\theta) \) such that \( \mathbf{q}_a \circ \Theta^{-1} - \tilde{h}_d(\theta) \equiv 0 \) is desired, the terms \( \tilde{h}_d(\theta), \partial \tilde{h}_d(\theta)/\partial \theta \), and \( \partial^2 \tilde{h}_d(\theta)/\partial \theta^2 \) may be found by sampling the limit cycle and using interpolation—without the need for curve fitting and differentiation.

**Remark 2:** For computational efficiency, the sampled functions \( \tilde{h}_d(\theta), \partial \tilde{h}_d(\theta)/\partial \theta \), and \( \partial^2 \tilde{h}_d(\theta)/\partial \theta^2 \) may be precomputed and stored in a lookup table.

**Remark 3:** This method is not equivalent to fitting \( \tilde{h}_d(\theta) \) to a set of splines, and then differentiating the splines to obtain \( \partial \tilde{h}_d(\theta)/\partial \theta \) and \( \partial^2 \tilde{h}_d(\theta)/\partial \theta^2 \). Cubic spline interpolation between sample points in conjunction with the method of Prop. 4 will result in estimates of \( \tilde{h}_d(\theta), \partial \tilde{h}_d(\theta)/\partial \theta \), and \( \partial^2 \tilde{h}_d(\theta)/\partial \theta^2 \) having an accuracy of \( \mathcal{O}(\tau^4) \), where \( \tau \) is the distance to the nearest sample point [16, Ch. 5]. Differentiation techniques would leave \( \tilde{h}_d(\theta) \) with an accuracy of \( \mathcal{O}(\tau^4) \), \( \partial \tilde{h}_d(\theta)/\partial \theta \) an accuracy of \( \mathcal{O}(\tau^4) \), and \( \partial^2 \tilde{h}_d(\theta)/\partial \theta^2 \) an accuracy of \( \mathcal{O}(\tau^2) \).

Because the controller (11) and (12) is being used, the sample-based virtual constraints will be continuous-phase invariant. It may be shown that sample-based virtual constraints are automatically invariant over the impact event. Thus, sample-based virtual constraints produce a valid HZD, and the analysis of Secs. II-E and II-F holds.

**B. Example 1: Enlarging the Basin of Attraction of a Stable, Passive Gait of a Two-link Biped**

The basin of attraction for the two-link passive biped depicted in Fig. 1 with parameters given in Tab. I walking on a ground slope of 0.02 rad (1.15 deg) is given in Fig. 2. The maximum coefficient of static friction at the stance leg end was assumed to be 0.6.

The steady-state passive gait, with the biped walking on a 0.02 rad slope, was enforced using a sample-based HZD controller. The inter-stride controller (11) and (12) was used with \( K_P = 200 \) and \( K_D = 25 \). The basin of attraction of the biped in closed loop with this controller is given Fig. 2. As an illustration, the closed-loop system was simulated for thirty steps with an initial condition \( x_0 = x_{0,\text{nom}} + \delta x_0 \), where \( x_{0,\text{nom}} \) is the state of the biped at the start of step on the periodic orbit of the passive gait and \( \delta x_0 = (0.2, 0.1, -1, 0)' \). Fig. 3 gives the evolution of the applied torque \( u \). Note that the relatively small peak control effort and that the control effort goes to zero as the state approaches the passive orbit.
C. Augmentation Functions

Consider the decomposition of $h_d$, into

$$h_d(s) = h_{d,0}(s) + h_{d,\delta}(s),$$

where $s \in [0, 1]$, $h_{d,0}$ is a nominal desired motion, and $h_{d,\delta} \in \mathcal{C}^2$ is an augmentation function. The function $h_{d,\delta}$ will be finitely parameterized and used to change the properties of the nominal motion associated with $h_{d,0}$. So that the analysis of Sec. II may be applied, the function $h_{d,\delta}$ is required to be such that $h_d$ satisfies the invariance of (26) under the impact map (3).

Let the augmentation function’s parameters be denoted by $a$. Then, augmenting the nominal motion with $h_{d,\delta}$, will result in the function $V_{\text{zero}}$ and the constant $\delta_{\text{zero}}$ being parameterized by $a$. The parameters $a$ can therefore be used to tune the restricted Poincaré map (19) so that its fixed point, the fixed point’s stability properties, and the lower bound may be selected. The approach is illustrated in the following example.

D. Example 2: Changing the minimum slope capability of a motion

For the two-link biped used in the example of Sec. III-B, the minimum ground slope for the sample-based HZD controller based upon the passive motion was found numerically to be 0.0171 rad (9.80 deg). Using numerical optimization, the augmentation function depicted in Fig. 4 was found such that the resulting closed-loop system was capable of walking a slope of $-0.01$ rad ($-0.523$ deg). As an illustration, the closed-loop system was simulated on zero slope, $\alpha = 0$, for an initial condition $x_0 = x_{0,\text{nom}} + \delta x_0$, where $x_{0,\text{nom}}$ is the state of the biped at the start of step on the periodic orbit of the passive gait on the nominal slope, $\alpha = 0.02$ rad, and $\delta x_0 = (0.025, 0.0125, 3, 0)'$. Fig. 5 gives the evolution of the applied torque $u$. Note that the relatively small peak control effort.

IV. Conclusions

This paper presented a novel control design methodology, sample-based HZD control, that is able to enlarge the basins of attraction of the gaits of passive dynamic walkers. The control acts without the need for full actuation—no actuation is assumed between the robot and the ground. The notion of a constraint augmentation function was introduced. Constraint augmentation functions are finitely parameterized functions added to the nominal, sample-based constraint, that enable the kinematic and dynamic properties of the gait to be modified. The results were illustrated on two examples. Animations of the associated motions are available on the web [9].

A. Future work

In an application that is similar to the one presented here, walking motions will be designed by optimizing over torques instead of optimizing over the output functions. It is hypothesized that this will allow for more efficient computation of desirable closed-loop motions—joint motion will no longer need to be slaved to finitely parameterized functions, but rather to the motions they naturally achieve on the limit cycle with a finitely parameterized torque profile.

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REFERENCES


APPENDIX I

EQUATIONS OF MOTION FOR TWO-LINK BIPED

The symmetric mass-inertia matrix, $D$, is

$$D(q_1)_{1,1} = (l - l_c)^2 m + I$$

$$D(q_1)_{1,2} = m l (l - l_c) \cos(q_1) - (l - l_c)^2 m - I$$

$$D(q_1)_{2,2} = -2 m l (l - l_c) \cos(q_1) + (2 l_c^2 + l^2) - 2 l_c l) m + 2 I.$$ (27a-c)

The matrix of coriolis and centrifugal terms is

$$C(q, \dot{q})_{1,1} = 0$$

$$C(q, \dot{q})_{1,2} = -m l \sin(q_1)(l - l_c) \dot{q}_2$$

$$C(q, \dot{q})_{2,1} = -m l \sin(q_1)(l - l_c) (\dot{q}_1 - \dot{q}_2)$$

$$C(q, \dot{q})_{2,2} = m l \sin(q_1)(l - l_c) \dot{q}_1.$$ (28a-d)

The gravity vector is

$$G(q)_1 = m g_0 \sin(q_1 - q_2)(l - l_c)$$

$$G(q)_2 = m g_0 [l_c (l - l_c) \sin(q_1 - q_2) - \sin(q_2)(l_c + l)].$$ (29a-b)

The symmetric mass-inertia matrix, $D$, is

$$D(q_1)_{1,1} = (l - l_c)^2 m + I$$

$$D(q_1)_{1,2} = m l (l - l_c) \cos(q_1) - (l - l_c)^2 m - I$$

$$D(q_1)_{2,2} = -2 m l (l - l_c) \cos(q_1) + (2 l_c^2 + l^2) - 2 l_c l) m + 2 I.$$ (27a-c)

The matrix of coriolis and centrifugal terms is

$$C(q, \dot{q})_{1,1} = 0$$

$$C(q, \dot{q})_{1,2} = -m l \sin(q_1)(l - l_c) \dot{q}_2$$

$$C(q, \dot{q})_{2,1} = -m l \sin(q_1)(l - l_c) (\dot{q}_1 - \dot{q}_2)$$

$$C(q, \dot{q})_{2,2} = m l \sin(q_1)(l - l_c) \dot{q}_1.$$ (28a-d)

The gravity vector is

$$G(q)_1 = m g_0 \sin(q_1 - q_2)(l - l_c)$$

$$G(q)_2 = m g_0 [l_c (l - l_c) \sin(q_1 - q_2) - \sin(q_2)(l_c + l)].$$ (29a-b)